CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH CONDITIONAL MOMENTS OF ORDER STATISTICS

ABSTRACT

THESIS
SUBMITTED FOR THE DEGREE OF

Doctor of Philosophy

IN

STATISTICS

BY

HASEEB ATHAR

Under the Supervision of

DR. MOHD. YAQUB

DEPARTMENT OF STATISTICS
&
OPERATIONS RESEARCH
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

2000

The work contained in this thesis is spread over in five chapters. A comprehensive bibliography has also been given at the end, which have been referred during our research.

Chapter I, is expository in nature and provides a brief review of the concepts and results concerning order statistics. Some continuous distributions, which are characterized, are also discussed here.

In Chapter II, a general class of distribution function $F(x) = ah(x) + b$ has been characterized through conditional expectation of a function of order statistic, conditioned on two order statistics and then result is expressed in terms of weighted means of function of conditioned order statistics. Further, some of its important deductions have also been discussed and examples based on this distribution have been listed. The specific distributions considered as a particular case of the general class of distribution are power function, Pareto, Weibull, inverse Weibull, beta of the first kind, beta of second the kind, extreme value, Cauchy, Gumbel, Burr type II, III, IV, V, VI, VII, VIII, IX, X, XI, and XII.

Chapter III deals with characterization of distributions by conditioning on a pair of order statistics. Here a general form of distributions:

(i) $F(x) = 1 - [ah(x) + b]^c$

(ii) $F(x) = [ah(x) + b]^c$

(iii) $F(x) = 1 - be^{-ah(x)}$

(iv) $F(x) = be^{-ah(x)}$

which was considered by Khan and Abu-Salih (1989) have been characterized through conditional expectations, conditioned on two order statistics and several of its important deductions are discussed. Further, examples based on these distributions have also been listed.

In Chapter IV, a class of distribution $F(x) = ah(x) + b$, considered earlier in Chapter II has been characterized by considering conditional moments of function of two order statistics, conditioned on a pair of order statistics and its important deductions are also discussed.

In Chapter V, we have adopted a different approach and an attempt has been made to characterize

$$F(x) = ah(x) + b$$

combining the two general form of distributions, through the conditional moments

$$E[g(X_{r+1:n}) | X_{r:n} = x]$$

and $$E[g(X_{r:n}) | X_{r+1:n} = x]$$

where $g(x) = e^{-h(x)}$, a function of $h(x)$.

CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH CONDITIONAL MOMENTS OF ORDER STATISTICS

THESIS

SUBMITTED FOR THE DEGREE OF

Doctor of Philosophy

IN

STATISTICS

BY

HASEEB ATHAR

Under the Supervision of

DR. MOHD. YAQUB

DEPARTMENT OF STATISTICS
&
OPERATIONS RESEARCH
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

2000

DEPARTMENT OF
STATISTICS & OPERATIONS RESEARCH
ALIGARH MUSLIM UNIVERSITY
ALIGARH – 202 2002, INDIA

June 28, 2000

CERTIFICATE

This is to certify that the matter embodied in the thesis entitled "CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH CONDITIONAL MOMENTS OF ORDER STATISTICS" by **Mr. Haseeb Athar** for the award of the Degree of Doctor of Philosophy in Statistics of the Aligarh Muslim University, Aligarh, is a record of bonafied research work carried out by him under my supervision and guidance.

The thesis has, in my opinion, reached the standard fulfilling the requirements of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other University or Institution for the award of any other degree or diploma.

(Dr. Mohd. Yaqub)

CONTENTS

ACKNOWLEDGEMENT

I wish to place on record my deep sense of gratitude to Prof. Abdul Hamid Khan for diligently helping me to identify the topic of this research, sorting out all the initial problems and put the work on track. Without his kind help it may not have been possible for me to get work going.

I owe a debt of gratitude, in equal measure, to my supervisor Dr. M. Yaqub whose unfailing help, guidance and encouragement made the completion of this work possible. On each problem of understanding, analysis and presentation his kind guidance was available for me.

Dr. S.N. Alam and Dr. R.U. Khan have also been extremely kind and helpful to me in conducting periodic review of this work. I am very thankful to them for their valuable suggestions.

At some stage or the other I have benefited from the help, suggestions and encouragement from my teachers in the department. Prof. S. Rehman, Prof. S.U. Khan, Prof. M.Z. Khan (Chairman), Mr. H. M. Islam deserve a special mention.

I am grateful to my parents, brother, sister, wife and other relatives for their time to time encouragement.

I express my heartfelt thanks to the galaxy of my noble colleagues and friends, whose continuous help and co-operation made my venture, come to fruition.

I express my thanks to the all office, seminar and computing lab staffs of our department for their consistent help and co-operation.

Last but not the least I am also thankful to University Grant Commission and the Department of Statistics & Operations Research for providing the funds and other facilities, when needed.

Date: 7/7/2K

Place: Aligarh

(Haseeb Athar)

PREFACE

Order statistics has immense role in characterization problems. Characterization results are located on the borderline between probability theory and mathematical statistics, and utilize numerous classical tools of mathematical analysis. One may observe that for samples from an exponential distribution $X_{1:n}$ has the same distribution as $X_{1:1}$ except for the rescaling, and this is true for any sample size n. Thus the form of the survival curve is the same for any series system constructed using such i.i.d. exponential components. Is the exponential distribution the only distribution with this property? The answer is yes. In fact, under mild regularity conditions, we shall see that it is enough that the result holds for any value of n. So, it is clear that knowledge of the distribution $F_{1:1}$ completely determines the distribution $F_{i:n}$ for every i, n. It also completely determines the marginal and joint distributions of various linear function of order statistics.

The useful characterizations results are those which shed light on modeling consequences of certain distributional assumptions and those which have potential for development of hypothesis tests for model assumptions. There is vast literature available on moments of order statistics and characterization results using moments of order statistics (Arnold *et al.*, 1991).

Characterization theorems are the only methods, which allow us to avoid the subjective choice of F and lead to the accurate $F(x)$ through simple properties. Application of characterization theorems can be illustrated with the help of following examples:

Let a store service a community of n persons. These persons visit the store independently of each other and their actual times of entering the store have the same distribution. Therefore, the n individuals can be associated with n independent and identically distributed random variables, namely, with their random times $X_j, 1 \le j \le n$, of entering the store. The store owner evidently observes the X_j in an increasing order. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the successive arrivals at the store. Assume that the store owner observes that the intervals $X_{1:n}, X_{2:n} - X_{1:n}, \ldots, X_{n:n} - X_{n-1:n}$ are also independent. This fact implies that the common distribution $F(x)$ of the X_j is again exponential. The emphasis here is that we have again arrived at a single possibility for $F(x)$. This means that the advice to all shops, or big departmental stores, dependent on a single, well defined model, when a decision is to be made on the number of employees, availability of items, etc.

The work contained in this thesis is spread over in five chapters. A comprehensive bibliography has also been given at the end, which have been referred during our research.

Chapter I, is expository in nature and provides a brief review of the concepts and results concerning order statistics. Some continuous distributions, which are characterized, are also discussed here.

In Chapter II, a general class of distribution function $F(x) = ah(x) + b$ has been characterized through conditional expectation of a function of order statistic, conditioned on two order statistics and then result is expressed in terms of weighted means of function of conditioned order statistics. Further, some of its important deductions have also been discussed and examples based on this distribution have been listed. The specific distributions considered as a particular case of the general class of distribution are power function, Pareto, Weibull, inverse Weibull, beta of the first kind, beta of second the kind, extreme value, Cauchy, Gumbel, Burr type II, III, IV, V, VI, VII, VIII, IX, X, XI, and XII.

Chapter III deals with characterization of distributions by conditioning on a pair of order statistics. Here a general form of distributions:

(i) $F(x) = 1 - [ah(x) + b]^c$

(ii) $F(x) = [ah(x) + b]^c$

(iii) $F(x) = 1 - be^{-ah(x)}$

(iv) $F(x) = be^{-ah(x)}$

which was considered by Khan and Abu-Salih (1989) have been characterized through conditional expectations, conditioned on two order statistics and several of its important deductions are discussed. Further, examples based on these distributions have also been listed.

In Chapter IV, a class of distribution $F(x) = ah(x) + b$, considered earlier in Chapter II has been characterized by considering conditional moments of function of two order statistics, conditioned on a pair of order statistics and its important deductions are also discussed.

In Chapter V, we have adopted a different approach and an attempt has been made to characterize

$$F(x) = ah(x) + b$$

combining the two general form of distributions, through the conditional moments

$$E[g(X_{r+1:n}) | X_{r:n} = x]$$

and $E[g(X_{r:n}) | X_{r+1:n} = x]$

where $g(x) = e^{-h(x)}$, a function of $h(x)$.

Chapter I

PRILIMINARIES AND BASIC CONCEPTS

1.1 Introduction:

In this chapter a brief review of the concepts and results used in subsequent chapters have been presented. Section 1.2 deals with some basic definition and applications of order statistics. In Section 1.3, some relevant results on recurrence relations and identities of order statistics are given. Section 1.4 is devoted to basic distribution theory of order statistics, whereas in Section 1.5, truncated and conditional distributions of order statistics have been discussed. In Section 1.6, references regarding characterization through conditional moments of order statistics are given and in Section 1.7, some basic continuous distributions are discussed.

1.2 Order statistics:

Let $X_1, X_2, ..., X_n$, a random sample of size n from a specified or unspecified population, be arranged in ascending order of magnitude so that

$$X_{1:n} \le X_{2:n} \le ..... \le X_{r:n} \le ..... \le X_{n:n}$$

then $X_{1:n}, X_{2:n}, ..., X_{n:n}$ are collectively called the order statistics of the sample and $X_{r:n}\,(r = 1, 2, ..., n)$ is called the r^{th} order statistic of the sample. Also $X_{1:n} = \min(X_1, X_2, ..., X_n)$ and $X_{n:n} = \max(X_1, X_2, ..., X_n)$ are called extreme order statistics or the smallest and the largest order statistics.

Order statistics is that branch of the subject of statistics, which deals with the mathematical properties of order statistics and with statistical methods based upon them. There is vast literature on order statistics. Developments through the early 1960s were synthesized in the volume edited by Sarhan and Greenberg (1962a), which, because of its numerous tables, retains its usefulness even today. Harter (1978-1992) has prepared an eight-volume annotated bibliography with an access of 4700 entries. David's (1981) book is perhaps the first book on order statistics dealing in detail with its different aspects. Asymptotic theory of extremes and related developments of order statistics are well described in an applausive work of Galamboas (1978, 1987) and on the more applied side, Gumbel's (1958) account continues to be valuable. Also, references may be made to Barnett and Lewis (1984), Balakrishnan and Cohen (1991), Arnold *et al.* (1992) and the references therein.

Order statistics and function of order statistics play an important role in statistical theory and methodology. Floods and droughts, longevity, breaking strength, aeronautics, oceanography, duration of human lives, organism, components and devices of various kinds can all be studied by the theory of extreme values. The range is widely used, particularly in statistical quality control, as an estimate of σ. Many short-cut test have been based on the range and other order statistics. In dealing with small samples, the studentized range is useful in a variety of ways. Apart from supplying the basis of many of quick tests, it plays a key role in procedures for ranking "treatment" means in the analysis of variance situation. The studentized range is also used in the detection of outliers.

By applying the Gauss-Markoff theorem of least squares, it is possible to use linear function of order statistics for estimating the parameters of distribution functions. This application is very useful, particularly when some of the observations in the sample have been "censored", since in that case standard methods of estimation tend to become laborious or otherwise unsatisfactory (Lloyd, 1952). Life tests provide an ideal illustration of the advantages of order statistics in censored data. Since such an experiment may take a long time to complete, it is often advantageous to stop after failure of the first r out of n similar items under test.

1.3 Recurrence relations and identities of order statistics:

Order statistics and their moments have received attention and interest from the beginning of this century since Galton (1902) and Pearson (1902) studied the distribution of the difference of two successive order statistics. Moments of order statistics are of immense importance in the statistical literature and have been numerically tabulated extensively for several distributions. For example, one can refer to David (1981), Arnold and Balakrishnan (1989), Arnold *et al.* (1992) for a detailed list of these tables. Recurrence relations of order statistics reduce the amount of direct computation and hence reduce the time and labour. To reduce the amount of direct computation, many authors, including Jones

(1948), Godwin (1949), Balakrishnan and Joshi (1981), Balakrishnan and Malik (1986), Khan *et al.* (1983 a, b), Khan and Khan (1987), Balakrishnan *et al.* (1988) and Sillitto (1951, 1964) carried out independent investigations satisfied by the moments of order statistics. Many of these relations and identities are quite useful as they express the higher order moments in terms of the lower order moments, thus making the evaluation of higher order moments easy, and in addition, provide some simple checks to test the accuracy of computation of moments of order statistics. Govindarajulu (1963), Arnold and Balakrishnan (1989) nicely summarized all these results and established some more recurrence relations and identities satisfied by single and product moments of order statistics. They then systematically applied these results in order to determine the maximum number of single and double integrals to be evaluated for the calculation of means, variances and co-variances of order statistics in a sample of size n, assuming the quantities for all sample sizes less than n are known. By a simple generalization of one of the results of Govindarajulu (1963), Joshi (1971) determined that for distributions symmetric about zero the number of double integrals to be evaluated for even values of n is in fact zero. Joshi and Balakrishnan (1982) established similar results for any arbitrary continuous distribution and applied them to improve over the bounds of Govindarajulu (1963). Ali and Khan (1997) obtained recurrence relations for expectations of a function of single order statistic from general class of distributions. Further, Ali and Khan (1998) established recurrence relations between expectation of function of two order statistics and then used these results to establish the recurrence relations between product moments, joint moment generating functions, quasi-ranges, characteristic functions for some class of doubly truncated and non-truncated distributions.

Order statistics play an important role in statistics to characterize the probability distributions. The recurrence relations for single and product moments of order statistics obtained by Khan *et al.* (1983a, b) have nicely been applied to characterize distributions (Khan and Khan, 1987; Khan and Ali, 1987). References may be made to Huang (1989), Khan and Abu-Salih (1988, 1989), Govindarajulu (1975), Lin (1988, 1989), Hwang and Lin (1984). Kamps (1991), Khan and Abuammoh (2000) and others.

1.4 Distribution of order statistics:

Here in this section we will discuss the basic distribution theory of order statistics by assuming that population is absolutely continuous. Let us assume that $X_1, X_2, \ldots\ldots, X_n$ is a random sample from an absolutely continuous population with probability density function (pdf) $f(x)$ and cumulative distribution function cdf $F(x)$. Further, let $X_{1:n} \le X_{2:n} \le \ldots\ldots \le X_{n:n}$ be the order statistics obtained by arranging the preceding random sample in increasing order of magnitude. Consider the event $x < X_{r:n} \le x + \delta x$, where δx is a small positive increment and we have

$$P(x < X_{r:n} \le x + \delta x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1 - F(x+\delta x)]^{n-r}$$
$$[F(x+\delta x) - F(x)] + O((\delta x)^2) \qquad (1.4.1)$$

where $O((\delta x)^2)$, a term of order $(\delta x)^2$, is the probability corresponding to the event having more than one X_i in the interval $(x, x+\delta x]$. From (1.4.1) we have the *pdf* of $X_{r:n}$ $(1 \le x \le n)$ as

$$f_{r:n}(x) = \lim_{\delta x \to 0} \left\{ \frac{P(x < X_{r:n} \le x + \delta x)}{\delta x} \right\}$$

$$= \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x) \ ; -\infty < x < \infty \tag{1.4.2}$$

The *pdf's* of smallest and largest order statistics follows from (1.4.2) at $r = 1$ and n respectively, i.e.,

$$f_{1:n}(x) = n[1-F(x)]^{n-1} f(x) \ ; \ -\infty < x < \infty \tag{1.4.3}$$

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x) \ ; \ -\infty < x < \infty \tag{1.4.4}$$

The cumulative distribution functions of the smallest and the largest order statistics are easily derived by integrating the *pdf's* in (1.4.3) and (1.4.4) and are given by:

$$F_{1:n}(x) = 1 - [1-F(x)]^n \ ; \ -\infty < x < \infty \tag{1.4.5}$$

and $$F_{n:n}(x) = [F(x)]^n \ ; \ -\infty < x < \infty \tag{1.4.6}$$

In general, the *cdf* of $X_{r:n}$ is given by

$$F_{r:n}(x) = P(X_{r:n} \le x)$$

$$= P(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} P(\text{exactly i of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1-F(x)]^{n-i} \ ; \ -\infty < x < \infty \tag{1.4.7}$$

The *cdf* of $X_{r:n}$ may also be obtained by integrating the *pdf* of $X_{r:n}$ as given by (1.4.2) so that,

$$F_{r:n}(x) = \int_{-\infty}^{x} f_{r:n}(t)dt$$

$$= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{x} [F(t)]^{r-1}[1-F(t)]^{n-r} f(t)dt$$

$$= \frac{n!}{(r-1)!(n-r)!} \int_{0}^{F(x)} u^{r-1}(1-u)^{n-r} du$$

$$= \mathrm{I}_{F(x)}(r, n-r+1) \tag{1.4.8}$$

The R.H.S. of (1.4.8) is Incomplete Beta Function as defined by Pearson (1934).

From the density function given by (1.4.2), we obtain the *kth* moment of $X_{r:n}$ to be:

$$\alpha_{r:n}^{(k)} = E[X_{r:n}^{k}] = \int_{-\infty}^{\infty} x^{k} f_{r:n}(x)dx \tag{1.4.9}$$

To derive the joint density function of two order statistics $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$, let us consider the event

$$(x < X_{r:n} \le x + \delta x, y < X_{s:n} \le y + \delta y) \;;\; -\infty < x < y < \infty$$

For small positive δx and δy we may write

$$P(x < X_{r:n} \le x + \delta x, y < X_{s:n} \le y + \delta y)$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}[F(y) - F(x+\delta x)]^{s-r-1}$$

$$[1-F(y+\delta y)]^{n-s}[F(x+\delta x) - F(x)][F(y+\delta y) - Fy)] +$$

$$O((\delta x)^2 \delta y) + O(\delta x(\delta y)^2) \tag{1.4.10}$$

where $O((\delta x)^2 \delta y)$ and $O(\delta x(\delta y)^2)$ are higher order terms which correspond to the probabilities of the event having more than one X_i in the interval $(x, x+\delta x]$ and at least one X_i in the interval $(y, y+\delta y]$, and of the event of having one X_i in $(x, x+\delta x]$ and more than one X_i in $(y, y+\delta y]$ respectively.

From (1.4.10) the joint *pdf* of $X_{r:n}$ and $X_{s:n}$ is given by:

$$f_{r,s:n}(x,y) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \left\{ \frac{P(x < X_{r:n} \le x+\delta x, y < X_{s:n} \le y+\delta y)}{\delta x \, \delta y} \right\}$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} [F(x)]^{r-1} [F(y)-F(x)]^{s-r-1}$$

$$[1-F(y)]^{n-s} f(x) f(y) \; ; -\infty < x < y < \infty \qquad (1.4.11)$$

The joint *pdf* of the smallest and the largest order statistics is obtained on putting $r=1$, $s=n$ in (1.4.11) and is given by:

$$f_{1,n:n}(x,y) = n(n-1)[F(y)-F(x)]^{n-2} f(x) f(y) \; ; -\infty < x < y < \infty \qquad (1.4.12)$$

Similarly, by setting $s=r+1$ in (1.4.11), we obtain the joint *pdf* of two consecutive order statistics, $X_{r:n}$ and $X_{r+1:n}$ $(1 \le r \le n-1)$, to be

$$f_{r,r+1:n}(x,y) = \frac{n!}{(r-1)!(n-r-1)!} [F(x)]^{r-1} [1-F(y)]^{n-r-1}$$

$$f(x) f(y) \; ; -\infty < x < y < \infty \qquad (1.4.13)$$

The joint cumulative distribution functions of $X_{r:n}$ and $X_{s:n}$, $(1 \le r < s \le n)$ can be obtained as follows:

$$F_{r,s:n}(x,y) = P(X_{r:n} \le x, X_{s:n} \le y)$$

$= P(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } x$
$\text{and at least } s \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } y)$

$= \sum_{j=s}^{n} \sum_{i=r}^{j} P(\text{exactly } i \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } x$
$\text{and exactly } j \text{ of } X_1, X_2, \ldots, X_n \text{ are at most y})$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!} [F(x)]^i [F(y) - F(x)]^{j-i} [1 - F(y)]^{n-j} \tag{1.4.14}$$

We can write the joint *cdf* of $X_{r:n}$ and $X_{s:n}$ in (1.4.14) equivalently as:

$$F_{r,s:n}(x, y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^{F(x)} \int_0^{F(y)} u^{r-1} (v-u)^{s-r-1} (1-v)^{n-s} \, du dv \tag{1.4.15}$$

$$= I_{F(x),F(y)}(r, s-r, n-s+1) \; ; -\infty < x < y < \infty \tag{1.4.16}$$

which is Incomplete Bivariate Beta Function.

It may be noted that for $x \geq y$

$$F_{r,s:n}(x, y) = F_{s:n}(y) \tag{1.4.17}$$

The product moments of j^{th} and k^{th} order of $X_{r:n}$ and $X_{s:n}$ respectively, $(1 \leq r < s \leq n)$ is given by:

$$\alpha_{r,s:n}^{(j,k)} = E[X_{r:n}^{j} \, X_{s:n}^{k}] = \iint\limits_{-\infty<x<y<\infty} x^j \; y^k f_{r,s:n}(x, y) dx dy \tag{1.4.18}$$

In general, the joint *pdf* of $X_{i_1:n}, X_{i_2:n}, \ldots, X_{i_k:n}$ for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ is given by

$$f_{i_1,i_2,\ldots,i_k:n}(x_{i_1:n},x_{i_2:n},\ldots,x_{i_k:n})$$
$$=n!\left\{\prod_{j=1}^{k} f(x_{i_j})\right\}\prod_{j=0}^{k}\left\{\frac{[F(x_{i_{j+1}})-F(x_{i_j})]^{i_{j+1}-i_j-1}}{(i_{j+1}-i_j-1)!}\right\}$$
$$;-\infty < x_{i_1} < x_{i_2} < \ldots < x_{i_k} < \infty \qquad (1.4.19)$$

where $x_0=-\infty, x_{k+1}=+\infty, i_0=0, i_{k+1}=n+1$

The distribution function of $X_{r:n}$ can also be expressed in terms of negative binomial probabilities instead of the binomial given in (1.4.7) as suggested by Khan, (1991):

$$F_{r:n}(x)=\sum_{i=0}^{n-r}\binom{n-1-i}{r-1}[F(x)]^r[1-F(x)]^{n-r-i}\,;\,-\infty<x<\infty$$

Remarks:

1. The ranking of random variables $X_1, X_2, \ldots, X_n$ is preserved under any monotonic increasing transformation of the random variables.

2. Regarding the probability integral transformation, if $X_{r:n}, 1 \le r \le n$, are the order statistics from a continuous distribution $F(x)$, then the transformation $U_{r:n}=F(X_{r:n})$ produces a random variable which is the r^{th} order statistics from a uniform distribution on (0,1)

3. Even if $X_1, X_2, \ldots, X_n$ are independent random variables, order statistics are not independent random variables.

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a continuous distribution. Then the set of order statistics

$\{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\}$ is both sufficient and complete (Lehmann, 1959).

5. Let X be a continuous random variable with $E[X_{r:n}] = \alpha_{r:n}$,

   a) $\alpha_{r:n}$ exists provided α exists, but converse is not necessarily true. Specially, if α does not exist, $\alpha_{r:n}$ may exist for certain (but not all) values of r.

   b) $\alpha_{r:n}$ for all n determine the distribution completely.

1.5 Truncated and conditional distribution of order statistics:

Let X be a continuous random variable having *pdf* $f(x)$ and *cdf* $F(x)$ in the interval $[-\infty, \infty]$.

Let $\int_{-\infty}^{Q_1} f(x)dx = Q$ and $\int_{-\infty}^{P_1} f(x)dx = P$ (1.5.1)

where Q_1 and P_1 are known constants. Then doubly truncated *pdf* of X is given by:

$$\frac{f(x)}{P-Q}\,;\ x \in (Q_1, P_1) \tag{1.5.2}$$

and the corresponding *cdf* is given by

$$\frac{F(x)-Q}{P-Q}\,;\ x \in (Q_1, P_1) \tag{1.5.3}$$

The lower and upper truncation points are Q_1, P_1 respectively; the degrees of truncation are Q (from below) and $1-P$ (from above). If we put $Q = 0$, the distribution will be truncated to the right. Similarly, for $P = 1$, the distribution will be truncated to the left.

Whereas for $Q=0, P=1$, we get the non truncated distribution. Truncated distributions are useful in finding the conditional distributions of order statistics.

In the following, we will relate the conditional distribution of order statistics (conditioned on another order statistic) to the distribution of order statistics from a population whose distribution is truncated from the original population distribution $F(x)$.

Statement 1.5.1 (David, 1981): Let $X_1, X_2, ..., X_n$ be a random sample from an absolutely continuous population with *cdf* $F(x)$ and probability density function $(pdf)\, f(x)$, and let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$, given that $X_{r:n} = x$ for $r < s$, is the same as the distribution of the $(s-r)$th order statistic obtained from a sample of size $(n-r)$ from a population whose distribution is truncated on the left at x.

Statement 1.5.2 (David, 1981): Let $X_1, X_2, ..., X_n$ be a random sample from an absolutely continuous population with *cdf* $F(x)$ and probability density function $(pdf)\, f(x)$, and let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{r:n}$, given that $X_{s:n} = y$ for $s > r$, is the same as the distribution of the rth order statistic obtained from a sample of size $(s-1)$ from a population whose distribution is truncated on the right at y.

Statement 1.5.3: Let $X_1, X_2, ..., X_n$ be a random sample from an absolutely continuous population with *cdf* $F(x)$ and probability density function $(pdf)\, f(x)$, and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$, given that $X_{r:n} = x$ and $X_{k:n} = z$ for $1 \leq r < s < k \leq n$, is the same as the distribution of the $(s-r)$th order statistic obtained from a sample of size $(k-r-1)$ from a population whose distribution is truncated on the left at x and on the right at z.

Proof: From (1.4.19), we can show that the joint density function of $X_{r:n}, X_{s:n}$, and $X_{k:n}$ $(1 \leq r < s < k \leq n)$ is given by

$$f_{r,s,k:n}(x,y,z) = \frac{n!}{(r-1)!(s-r-1)!(k-s-1)!(n-k)!}[F(x)]^{r-1}$$
$$[F(y)-F(x)]^{s-r-1}[F(z)-F(y)]^{k-s-1}[1-F(z)]^{n-k}$$
$$f(x)f(y)f(z); \ -\infty < x < y < z < \infty \qquad (1.5.4)$$

From equations (1.5.4) and (1.4.11), we obtain the conditional density function of $X_{s:n}$ given that $X_{r:n} = x$ and $X_{k:n} = z$ is as

$$f_{s:n}(y \mid X_{r:n} = x, X_{k:n} = z) = \frac{f_{r,s,k:n}(x,y,z)}{f_{r,k:n}(x,z)}$$

$$= \frac{(k-r-1)!}{(s-r-1)!(k-s-1)!}\left[\frac{F(y)-F(x)}{F(z)-F(x)}\right]^{s-r-1}$$
$$\left[\frac{F(z)-F(y)}{F(z)-F(x)}\right]^{k-s-1}\frac{f(y)}{F(z)-F(x)}; \quad x<y<z \qquad (1.5.5)$$

The result follows immediately from (1.5.5) after noting that $\frac{F(y)-F(x)}{F(z)-F(x)}$ and $\frac{f(y)}{F(z)-F(x)}$ are the *cdf* and density function of the population whose distribution is obtained by truncating the distribution $F(x)$ on the left at x and on the right at z.

Remark 1.5.1: Statement 1.5.1 follows by replacing k with $n+1$ and $F(z)$ with 1 in equation (1.5.5).

Remark 1.5.2: Statement 1.5.2 follows by letting $r=0$ and $F(x)=0$ in the equation (1.5.5).

1.6 Characterizations through conditional moments of order statistics:

Many authors have studied characterization through conditional moments of order statistics. Shanbhag (1970) characterized the exponential and geometric distribution in terms of conditional expectation. His result for exponential distribution is further generalized by Hamdan (1972). Beg and Kirmani (1978) showed that the conditional variance of $X_{r+1:n}$ given $X_{r:n}=x$ does not depend on x if and only if X has exponential distribution. Khan and Beg (1987) extended the result of Beg and Kirmani (1978) and showed that variance of $X_{r+1:n}^{p}$ given $X_{r:n}=x$ does not depend on x if and only if X has Weibull distribution for $p>0$. Khan and Khan (1987) obtained recurrence relations for single and product moments of order statistics from doubly truncated Burr distribution (Burr type XII) utilizing the results developed by Khan *et al.* (1983 a, b) and then used some results to characterize the Burr distribution. Khan and Abu Salih (1988) characterized the Weibul and inverse Weibull distribution for double order gap. Khan and Ali (1987) used the conditional moments of order statistics with higher order gap to characterize Weibull, exponential, Rayleigh, Burr, Log logistic, Pareto and power function distributions. Khan *et al.* (1988) obtained the moments of order statistics from a symmetrically truncated Logistic distribution and used the result to characterize the Logistic distribution.

Talwalkar (1977) obtained characterization results for the general class of absolutely continuous distributions in terms of conditional expectations. On the basis of certain generalizations of Talwalker's result, Ouyang (1983) characterized a general class of continuous distribution including the power, Weibull, beta distributions and also two forms of Burr's distribution.

Ferguson (1967), Kotlarski (1972), Khan and Beg (1987), Khan and Khan (1986, 1987), Pakes *et al.* (1996) and others characterize some specific distributions through conditional moments of order statistics by expressing

$$\xi_r(x) = E[h(X_{r+1:n}) | X_{r:n} = x]$$

and $\bar{\xi}_r(x) = E[h(X_{r:n}) | X_{r+1:n} = x]$

as $ah(x)+b$ with proper choice of a, b and $h(x)$.

Khan and Abu-Salih (1989) characterized some general form of distribution through $\xi_r(x)$ and $\bar{\xi}_r(x)$ by properly choosing a, b and $h(x)$. Some related results were also obtained by Ouyang (1995), Dimaki and Xekalaki (1996), Franco and Ruiz (1995, 1999). Reference may also be made to the Editorial note $Sankhy\bar{a}, Ser. A, 60(1998), 150$.

Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) where the conditioned order statistic may not be adjacent one. Other related references are Wu and Ouyang (1996), Blaquez and Rebollo (1997) and Franco and Ruiz (1997).

Further, Khan and Athar (2000a) characterized the general form of distributions considered by Khan and Abu-Salih (1989) through conditional expectations, conditioned on two order statistics and listed about fifty examples, which are given in Chapter III.

Balasubramanian and Beg (1992) characterized some distributions through the relation

$$E[h(X) \mid x \leq X \leq y] = \frac{h(x) + h(y)}{2}$$

where $h(x)$ is a measurable function of x.

Khan and Athar (2000b) extended the result of Balasubramanian and Beg (1992) to order statistics and characterized a general class of distribution $F(x) = [ah(x) + b]$ through conditional expectation of a function of order statistic, conditioned on two order statistics and expressed in terms of weighted mean of function of conditioned order statistics. (Please refer to Chapter II). Further, Khan and Athar (2000c) established a characterization theorem for the general class of distribution through conditional expectation of function of two order statistics. The result is given in Chapter IV.

1.7 Some continuous distributions:

I. Pareto Distribution:

A random variable X is said to have the Pareto distribution if its probability density function $(pdf)\, f(x)$ and distribution function $(df)\, F(x)$ are of the form given below:

$$f(x) = p\lambda^p x^{-(p+1)}; \qquad \lambda \leq x < \infty; \quad \lambda, p > 0$$

$$F(x) = 1 - \lambda^p x^{-p}; \qquad \lambda \leq x < \infty; \quad \lambda, p > 0$$

Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc.

II. Power Function Distribution:

A random variable X is said to have a power function distribution if its probability density function (pdf) and distribution function (df) are of the form given below:

$$f(x) = p\lambda^{-p}x^{p-1}; \qquad 0 \le x < \lambda; \quad \lambda, p > 0$$

$$F(x) = \lambda^{-p}x^{p}; \quad 0 \le x < \lambda; \quad \lambda, p > 0$$

The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if X has a Power function distribution, then $Y = \frac{1}{X}$ has a Pareto distribution.

III. Beta Distribution:

a) Beta distribution of First Kind:

A random variable X is said to have the beta distribution of first kind if its probability density function (pdf) is of the form:

$$f(x) = \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1}; \; 0 \le x \le 1, \; p, q > 0$$

beta distribution arises as the distribution of an ordered variable from a rectangular distribution. Suppose $X_{r:n}$ is an ordered sample from $U(0,1)$, then $X_{r:n} \sim B(r, n-r+1)$. The standard rectangular distribution $R(0,1)$ is the special case of beta distribution of first kind obtained by putting the exponents p and q equal to 1. If $q=1$, the distribution reduces to power function distribution.

b) Beta distribution of Second Kind:

The continuous random variable X which is distributed according to probability law:

$$f(x)=\frac{1}{B(p,q)}\frac{x^{p-1}}{(1+x)^{p+q}} \qquad (p,q)>0,\ 0\le x<\infty$$

is known as a beta variate of the second kind with parameters p and q.

Remark 1.7.1: Beta distribution of second kind can be transformed to beta distribution of first kind by transformation $1+x=\frac{1}{y}$.

Usage: The Beta distribution is one of the most frequently employed distributions to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with "uncertain" transition probabilities.

IV. Weibull Distribution:

A random variable X is said to have a Weibull distribution if its probability density function (pdf) is given by:

$$f(x)=\theta\, p\, x^{p-1}\, e^{-\theta\, x^{p}}; \qquad 0\le x<\infty;\ \theta>0,\ p>0$$

and the cumulative distribution function (cdf) is given by

$$F(x)=1-e^{-\theta\, x^{p}}; \qquad 0\le x<\infty;\ \theta>0,\ p>0$$

Remark 1.7.2: If we put $p=1$ in Weibull distribution, we get the *pdf* of exponential distribution.

Remark 1.7.3: If we put $p=2$, it gives *pdf* of Rayleigh distribution.

Remark 1.7.4: If X has a Weibull distribution, then the probability density function of $Y=-p\log\left(\frac{X}{\alpha}\right)$ is:

$$f(y)=e^{-y}e^{-e^{-y}}$$

Which is a form of the **Extreme Value distribution.**

Remark 1.7.5: Probability density function (pdf) and the cumulative distribution function (cdf) of **inverse Weibull distribution** is given by:

$$f(x)=\theta\, p\, x^{-(p+1)}\, e^{-\theta\, x^{-p}}, \qquad 0\le x<\infty;\ \theta>0,\ p>0$$

$$F(x)=e^{-\theta\, x^{-p}}, \qquad 0\le x<\infty;\ \theta>0,\ p>0$$

Usage: Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

V. Exponential Distribution:

A random variable X is said to have an exponential distribution if its probability density function (pdf) is given by:

$$f(x)=\theta e^{-\theta x}; \quad 0 \le x < \infty;\ \theta > 0$$

and the *cdf* is given by

$$F(x)=1-e^{-\theta x}; \quad 0 \le x < \infty;\ \theta > 0$$

Usage: The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable X assuming non-negative values satisfies the assumption,

$$P(X > s+t \mid X > s) = P(X > t) \text{ for all } s \text{ and } t,$$

then X will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on past, for example, in studying the life of a bulb etc.

VI. Generalized Linear Exponential Distribution:

A random variable X is said to have a generalized linear exponential distribution if its probability density function (pdf) is given by:

$$f(x)=(\lambda+\theta\, p\, x^{p-1})e^{-(\lambda x+\theta x^{p})}; \quad 0 \le x < \infty;\ \lambda,\theta,p>0$$

and the *cdf* is given by

$$F(x)=1-e^{-(\lambda x+\theta x^{p})}; \quad 0 \le x < \infty;\ \lambda,\theta,p>0$$

If we put $\lambda = 0$, it becomes Weibull distribution whereas for $\theta = 0$; and for $\lambda = 0, p = 2$, it reduces to Exponential and Rayleigh distributions respectively.

VII. Rectangular Distribution:

A random variable X is said to have a rectangular distribution if its probability density function (pdf) is given by:

$$f(x) = \frac{1}{\lambda - \beta}; \ \beta \le x \le \lambda$$

and the cdf is given by:

$$F(x) = \frac{x - \beta}{\lambda - \beta}; \ \beta \le x \le \lambda$$

The standard Rectangular distribution $R(0,1)$ is obtained by putting $\beta = 0$ and $\lambda = 1$. It is noted that every distribution function follows rectangular distribution $R(0,1)$. This distribution is used in "rounding off" errors, probability integral transformation, random number generation, traffic flow, generation of normal, exponential distribution etc.

VIII. Burr Distribution:

Let X be a continuous random variable, then different forms of cumulative distribution function of X are listed below (Johnson and Kotz, 1970):

I $\quad F(x) = x, \qquad 0 < x < 1$

II $\quad F(x) = (1 + e^{-x})^{-k}, \qquad -\infty < x < \infty$

III $F(x) = (1 + x^{-c})^{-k}, \qquad 0 \le x < \infty$

IV $F(x) = \left[1 + \left(\dfrac{c - x}{x}\right)^{1/c}\right]^{-k}, \qquad 0 \le k \le c$

V $F(x) = [1 + ce^{-\tan x}]^{-k}, \qquad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

VI $F(x) = [1 + ce^{-k \sinh x}]^{-k}, \qquad -\infty < x < \infty$

VII $F(x) = 2^{-k}(1 + \tanh x)^k, \qquad -\infty < x < \infty$

VIII $F(x) = \left(\dfrac{2}{\pi}\tan^{-1} e^x\right), \qquad -\infty < x < \infty$

IX $F(x) = 1 - \dfrac{2}{c[(1 + e^x)^k - 1] + 2}, \qquad -\infty < x < \infty$

X $F(x) = (1 + e^{-x^2})^k \qquad 0 \le x < \infty$

XI $F(x) = \left(x - \dfrac{1}{2\pi}\sin 2\pi x\right)^k \qquad 0 \le x \le 1$

XII $F(x) = 1 - (1 + x^c)^{-k} \qquad 0 \le x < \infty$

where k and c are positive parameters.

Special attention is given to type XII, which *pdf* is given as:

$$f(x) = kcx^{c-1}(1 + x^c)^{-(k+1)};\ 0 \le x < \infty;\ k, c > 0$$

This distribution is frequently used for the purpose of graduation and in reliability theory. At $c = 1$, it is called **Lomax distribution** whereas for $k = 1$, it is known as **Log-logistic distribution**.

IX. Cauchy Distribution:

The special form of the Pearson Type VII distribution, with probability density function (pdf):

$$f(x) = \frac{1}{\pi\lambda} \frac{1}{[1 + \{(x-\theta)/\lambda\}^2]} \qquad -\infty < x < \infty;\ \lambda > 0;\ -\infty < \theta < \infty$$

is called the Cauchy distribution.

The cumulative distribution function (cdf) is given by:

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\lambda}\right) \qquad -\infty < x < \infty;\ \lambda > 0;\ -\infty < \theta < \infty$$

The distribution is symmetrical about $x = \theta$. The distribution does not possess finite moments of order greater than or equal to 1, and so does not possess a finite expected value or standard deviation. However, θ and λ are location and scale parameters, respectively, and may be regarded as being analogous to mean and standard deviation.

There is no standard form of the Cauchy distribution, as it is not possible to standardize without using (finite) values of mean and standard deviation, which does not exist in this case. However, a standard form is obtained by putting $\theta = 0, \lambda = 1$. The standard probability density function is given by:

$$f(x) = \frac{1}{\pi} \frac{1}{(1 + x^2)} \qquad -\infty < x < \infty$$

and the standard cumulative distribution function is

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x \qquad -\infty < x < \infty$$

Chapter II

CHARACTERIZATION OF DISTRIBUTIONS THROUGH ORDER STATISTICS

2.1 Introduction

Order statistics have been extensively used in problems on ranges, quasi-ranges, tolerance limits, estimation of parameters, censored samples, selection and ranking problems. Many recurrence relations between moments of order statistics are available in the literature. References may be made to Joshi (1971), Joshi and Balakrishnan (1982), Khan *et. al.* (1983a, b), Balakrishnan *et al.* (1988), Kamps (1991), Ali and Khan (1997, 1998) and references therein.

Here in this chapter a general class of distribution function $F(x) = ah(x) + b$ has been characterized through conditional expectation of a function of order statistic, conditioned on two order statistics and then the result is expressed in terms of weighted mean of function of conditioned order statistics. Further some of its important deductions have also been discussed here.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous population having probability density function $(pdf)\, f(x)$ and distribution function $(df)\, F(x)$, over the support

(α,β) and let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the corresponding order statistics. Balasubramanian and Beg (1992) used the relation

$$E[h(X) \mid x \le X \le y] = \frac{h(x)+h(y)}{2} \tag{2.1.1}$$

to characterize some distribution functions where $h(x)$ is a measurable function of x.

It may be seen that the conditional distribution of $X_{j:n}$ given $X_{r:n} = x$ and $X_{s:n} = y, 1 \le r < j < s \le n$, is unconditional distribution of $X_{j-r:s-r-1}$ truncated to the left at x and to the right at y. That is,

$$E[h(X_{j:n}) \mid X_{r:n} = x, X_{s:n} = y]$$
$$= E[h(X_{j-r:s-r-1}) \mid x \le X_{j-r:s-r-1} \le y] \tag{2.1.2}$$

and thus at $j = r+1$ and $s = r+2$,

$$E[h(X_{r+1:n}) \mid X_{r:n} = x, X_{r+2:n} = y] = E[h(X_{1:1}) \mid x \le X_{1:n} \le y]$$
$$= E[h(X) \mid x \le X \le y] \tag{2.1.3}$$

Further since

$$\sum_{r=1}^{n} E(X_{r:n}) = nE(X) \tag{2.1.4}$$

we have,

$$\frac{1}{s-r-1} \sum_{j=r+1}^{s-1} E[h(X_{j:n}) \mid X_{r:n} = x, X_{s:n} = y] = E[h(X) \mid x \le X \le y] \tag{2.1.5}$$

Therefore, result obtained by Balasubramanian and Beg (1992) in terms of (2.1.1) is also true for (2.1.5) for order statistics over summation. However, for the expression

$$E[h(X_{j:n}) \mid X_{r:n} = x, X_{s:n} = y] \tag{2.1.6}$$

perhaps no such results are available in the literature. We have, therefore, made an attempt to characterize a family of distributions utilizing conditional expectation (2.1.6).

The doubly truncated *pdf* of continuous random variable will be denoted as

$$\frac{f(x)}{P-Q}, Q_1 < x < P_1$$

where $F(Q_1) = Q, F(P_1) = P$

and the *df* is

$$\frac{F(x)-Q}{P-Q}$$

Also, we shall use the convention

$$X_{0:n} = Q_1 \text{ and } X_{n:n-1} = P_1$$

2.2 Characterization Theorem

Theorem 2.2.1: For any continuous and differentiable function $h()$ and $m = s-r-1, 1 \le r < s \le n, i = 1,2,\ldots,m$

$$E\left[h(X_{r+i:n}) \mid X_{r:n} = x, X_{s:n} = y\right] = \frac{(m-i+1)h(x) + ih(y)}{(m+1)} \tag{2.2.1}$$

if and only if

$$F(x) = ah(x) + b \ , \ \alpha \le x \le \beta \tag{2.2.2}$$

with $F(\alpha) = 0$ and $F(\beta) = 1$

Proof: To prove (2.2.2) implies (2.2.1), we have for $F(x) = ah(x) + b$ (Ali and Khan, 1997),

$$E[h(X_{i:m})] - E[h(X_{i-1:m})] = \frac{P-Q}{(m+1)a} \tag{2.2.3}$$

where $P = F(y), Q = F(x)$, and $P - Q = a[h(y) - h(x)]$.

Writing (2.2.3) recursively and noting that

$$E[h(X_{0:m})] = E[h(Q_1)] = h(x) \tag{2.2.4}$$

and $E[h(X_{r+i:n}) \mid X_{r:n} = x, X_{s:n} = y] = E[h(X_{i:m}) \mid x \le X_{i:m} \le y]$ (2.2.5)

the result follows.

To prove (2.2.1) implies (2.2.2), we have (Ali and Khan, 1997),

$$E[h(X_{i:m})] - E[h(X_{i-1:m})]$$
$$= \binom{m}{i-1} \int_x^y h'(t) \left[\frac{F(t)-F(x)}{F(y)-F(x)}\right]^{i-1} \left[\frac{F(y)-F(t)}{F(y)-F(x)}\right]^{m-i+1} dt$$
$$= \frac{F(y)-F(x)}{(m+1)a}$$

or, $$\frac{(m+1)!a}{(i-1)!(m-i+1)!} \int_x^y h'(t)[F(t)-F(x)]^{i-1}[F(y)-F(t)]^{m-i+1} dt = [F(y)-F(x)]^{m+1}$$

Differentiating both sides once w.r.t. x, we get

$$\frac{m!}{(i-2)!(m-i+1)!}a\int_x^y h'(t)[F(t)-F(x)]^{i-2}$$
$$[F(y)-F(t)]^{m-i+1}dt=[F(y)-F(x)]^m$$

Differentiating again both sides w.r.t x, we have

$$\frac{(m-1)!}{(i-3)!(m-i+1)!}a\int_x^y h'(t)[F(t)-F(x)]^{i-3}$$
$$[F(y)-F(t)]^{m-i+1}dt=[F(y)-F(x)]^{m-1}$$

.
.
.

Differentiation at the $(i-1)th$ times gives,

$$\frac{(m-i+2)!}{(m-i+1)!}a\int_x^y h'(t)[F(y)-F(t)]^{m-i+1}dt=[F(y)-F(x)]^{m-i+2}$$

Differentiating again w.r.t. x, we have

$$(m-i+2)ah'(x)[F(y)-F(x)]^{m-i+1}$$
$$=(m-i+2)[F(y)-F(x)]^{m-i+1}f(x)$$

$$\Rightarrow \quad ah'(x)=f(x)$$

i.e. $\quad F(x)=ah(x)+b$

where b statisfies initial conditions of a *d.f.* $F(x)$.

This proves the theorem.

Remark 2.2.1: At $i=1, s=r+2, m=s-r-1=1$,

$$E\left[h(X_{r+1:n})\middle| X_{r:n}=x, X_{r+2}=y\right]=E\left[h(X)\middle| x\le X\le y\right]$$
$$=\frac{h(x)+h(y)}{2}$$

and also in view of (2.1.5),

$$\frac{1}{(s-r-1)}\sum_{j=r+1}^{s-1} E\left[h(X_{j:n}) \mid X_{r:n}=x, X_{s:n}=y\right]=\frac{h(x)+h(y)}{2}$$

as obtained by Balasubramanian and Beg (1992).

Remark 2.2.2: At $s=n+1, X_{n+1:n}=y=\beta, m=n-r$

$$E\left[h(X_{r+i:n}) \mid X_{r:n}=x\right]=\frac{(n-r-i+1)h(x)+ih(\beta)}{(n-r+1)} \qquad (2.2.6)$$

as given by Franco and Ruiz (1997).

Further at $y=\beta$,

$$F(\beta)=ah(\beta)+b=1\ ,\ h(\beta)=\frac{1-b}{a}$$

Therefore r.h.s. of (2.2.6) is

$$\frac{(n-r-i+1)h(x)}{(n-r+1)}+\frac{i}{n-r+1}\frac{(1-b)}{a} \qquad (2.2.7)$$

if and only if

$$F(x)=-ah(x)+(1-b)$$

$$=1-\left[ah(x)+b\right]$$

as obtained by Khan and Abouammoh (2000). At $i=1$, the result was given by Khan and Abu-Salih(1989).

Remark 2.2.3: At $r=0, X_{0:n}=x=\alpha,\ m=s-1$

$$E\left[h(X_{i:n}) \mid X_{s:n}=y\right]=\frac{(s-i)h(\alpha)+ih(y)}{s}$$

as given by Franco and Ruiz (1997).

Now since $F(\alpha) = ah(\alpha) + b = 0$

Hence, $E[h(X_{i:n}) \mid X_{s:n} = y] = \frac{i}{s} h(y) - \frac{(s-i)}{s}\frac{b}{a}$

as obtained by Khan and Abouammoh (2000) at $c = 1$.

Further, at $s = i + 1$, the result was derived by Khan and Abu-Salih (1989).

Remark 2.2.4: Beg and Balasubramanian (1990) have characterized *df* $F(x)$ through

$$E\left[\frac{1}{s-1}\sum_{i=1}^{s-1} h(X_{i:n}) \mid X_{s:n} = x\right] = \frac{1}{2}(h(x) + h(a+))$$

using order statistics. But this could have been obtained rather easily without using order statistics simply by noting that

$$E\left[\sum_{i=1}^{s-1} h(X_{i:n}) \mid X_{s:n} = x\right] = E\left[\sum_{i=1}^{s-1} h(X_{i:s-1}) \mid X_{i:s-1} \le x\right]$$

$$= (s-1)\, E[h(X) \mid X \le x]$$

$$= (s-1) E[h(X_{1:n}) \mid X_{2:n} = y]$$

and therefore the result given by Beg and Balasubramanian (1990) can be obtained by putting $i = 1$ and $s = 2$ in Remark 2.2.3.

2.3 Examples

Proper choice of a, b and $h(x)$ characterize the distributions as given below:

Distributions	$F(x)$	a	b	$h(x)$
1. Power function	$a^{-p}x^{p}$, $0 \le x \le a$	a^{-p}	0	x^{p}
2. Pareto	$1-a^{p}x^{-p}$, $a \le x < \infty$	$-a^{p}$	1	x^{-p}
3. Weibull	$1-e^{-\theta x^{p}}$, $0 \le x < \infty$	-1	1	$e^{-\theta x^{p}}$
4. Inverse Weibull	$e^{-\theta x^{-p}}$, $0 \le x < \infty$	1	0	$e^{-\theta x^{-p}}$
5. Beta of first kind	$1-(1-x)^{p}$, $0 \le x \le 1$	-1	1	$(1-x)^{p}$
6. Beta of second kind	$1-(1+x)^{-1}$, $0 \le x \le 1$	-1	1	$(1+x)^{-1}$
7. Extreme value	$1-\exp(-e^{x})$, $-\infty < x < \infty$	-1	1	$\exp(-e^{x})$
8. Cauchy	$\frac{1}{\pi}\tan^{-1}x+\frac{1}{2}$, $-\infty < x < \infty$	$\frac{1}{\pi}$	$\frac{1}{2}$	$\tan^{-1}x$
9. Gumbel	$\exp(-e^{-x})$, $-\infty < x < \infty$	1	0	$\exp(-e^{-x})$
10. Burr type II	$(1+e^{-x})^{-k}$, $-\infty < x < \infty$	1	0	$(1+e^{-x})^{-k}$
11. Burr type III	$(1+x^{-c})^{-k}$, $0 \le x < \infty$	1	0	$(1+x^{-c})^{-k}$
12. Burr type IV	$\left[1+\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{-k}$, $0 \le x \le c$	1	0	$\left[1+\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{-k}$
13. Burr type V	$[1+ce^{-\tan x}]^{-k}$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	1	0	$[1+ce^{-\tan x}]^{-k}$
14. Burr type VI	$[1+ce^{-k\sinh x}]^{-k}$, $-\infty < x < \infty$	1	0	$[1+ce^{-k\sinh x}]^{-k}$

15. Burr type VII	$2^{-k}(1+\tanh x)^{k}$, $-\infty<x<\infty$	2^{-k}	0	$(1+\tanh x)^{k}$
16. Burr type VIII	$\left(\frac{2}{\pi}\tan^{-1}e^{x}\right)^{k}$, $-\infty<x<\infty$	$\left(\frac{2}{\pi}\right)^{k}$	0	$(\tan^{-1}e^{x})^{k}$
17. Burr type IX	$1-2[c\{(1+e^{x})^{k}-1\}+2]^{-1}$,$-\infty<x<\infty$	-2	1	$[c\{1+e^{x})^{k}-1\}+2]^{-1}$
18. Burr type X	$(1-e^{-x^{2}})^{k}$, $x\geq 0$	1	0	$(1-e^{-x^{2}})^{k}$
19. Burr type XI	$\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^{k}$, $0\leq x\leq 1$	1	0	$\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^{k}$
20. Burr type XII	$1-(1+\theta x^{p})^{-m}$, $x\geq 0$	-1	1	$(1+\theta x^{p})^{-m}$

Chapter III

Chapter III

ON CHARACTERIZATION OF DISTRIBUTIONS BY CONDITIONING ON A PAIR OF ORDER STATISTICS

3.1 Introduction

Moments of order statistics are extensively used in characterization of specific distributions. Here, in this chapter a general form of distributions considered by Khan and Abu-Salih (1989) have been characterized through conditional expectations, conditioned on two order statistics and several of its important deductions are discussed. To this end we proceed as follows:

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a continuous population having probability density function (*pdf*) $f(x)$ and distribution function (*df*) $F(x)$. Let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ be the corresponding order statistics.

Khan and Abu-Salih (1989) characterized general form of distributions:

$$\left.\begin{array}{l} \text{i) } F(x) = 1 - [ah(x) + b]^c \\ \text{ii) } F(x) = [ah(x) + b]^c \\ \text{iii) } F(x) = 1 - be^{-ah(x)} \\ \text{iv) } F(x) = be^{-ah(x)} \end{array}\right\} \tag{3.1.1}$$

through conditional expectation of functions of order statistics fixing adjacent order statistic.

Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) where the conditioned order statistic may not be adjacent one. Other related references are Wu and Ouyang (1996), Blaquez and Rebollo (1997) and Franco and Ruiz (1997).

In this chapter the conditional expectation of functions of order statistics

$$E[h(X_{j:n}) | X_{r:n} = x, X_{s:n} = y],\ 1 \le r < j < s \le n$$

conditioned on two order statistics have been considered to characterize the general form of distributions given in (1.1) at $j = r+1$ and $j = s-1$. It may be noted that, because of the Markovian properties of order statistics, it is of no use to condition it on more than two order statistics. Khan *et al.* (1988) were perhaps the first to characterize logistic distribution fixing two order statistics. Balasubramanian and Beg (1992) and Balasubramanian and Dey (1997) have also obtained some results fixing two order statistics, but our approach is entirely different to theirs.

To establish our results, we have utilized the relationship between conditional and truncated distributions.

The conditional distribution of $X_{j:n}$ given $X_{r:n} = x$ and $X_{s:n} = y,\ 1 \le r < j < s \le n$ is unconditional distribution of $X_{j-r:s-r-1}$ truncated to the left at x and to the right at y.

That is

$$E[h(X_{j:n}) | X_{r:n} = x, X_{s:n} = y] \\ = E[h(X_{j-r:s-r-1}) | x \le X_{j-r:s-r-1} \le y] \qquad (3.1.2)$$

and

$$E[h(X_{r+1:n})|\, X_{r:n}=x, X_{r+2}=y]=E[h(X)|\, x\le X\le y] \qquad (3.1.3)$$

The doubly truncated *pdf* of continuous random variable will be denoted as

$$\frac{f(x)}{P-Q},\ Q_1<x<P_1 \qquad (3.1.4)$$

where

$$F(Q_1)=\int_{-\infty}^{Q_1} f(x)\,dx=Q$$

$$F(P_1)=\int_{-\infty}^{P_1} f(x)\,dx=P$$

and the *df* is

$$\frac{F(x)-Q}{P-Q} \qquad (3.1.5)$$

Also, we will denote by convention

$$X_{0:n}=Q_1 \text{ and } X_{n:n-1}=P_1 \qquad (3.1.6)$$

3.2 Characterization Theorems

Theorem 3.2.1: For any continuous and differentiable function $h()$ and $1\le r<s\le n$,

$$E\left[h(X_{r+1:n})|\, X_{r:n}=x, X_{s:n}=y\right]$$

$$=(-1)^m P_2^m h(y)\prod_{i=1}^{m}\frac{ic}{(ic+1)}+Q_2 h(x)\sum_{i=0}^{m-1}(-1)^i P_2^i\prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1}$$

$$-\frac{b}{a}\sum_{i=0}^{m-1}(-1)^i P_2^i\frac{1}{(m-i)c}\prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1} \qquad (3.2.1)$$

if and only if

$$F(x) = 1 - [ah(x) + b]^c, \alpha \le x \le \beta \tag{3.2.2}$$

where $m = s - r - 1$, $P_2 = \dfrac{1-P}{P-Q}$ and $Q_2 = \dfrac{1-Q}{P-Q}$,

$a \neq 0, (m-i)c \neq 0, (m-i)c + 1 \neq 0$

Proof: First we will prove that (3.2.2) implies (3.2.1). We have (Ali and Khan, 1997),

$$E[h(X_{r:n})] - E[h(X_{r:n-1})] = \\ -\binom{n-1}{r-1}\int_{Q_1}^{P_1} h'(x)[F(x)]^r [1-F(x)]^{n-r}\, dx \tag{3.2.3}$$

For doubly truncated distribution function (3.2.2) at $Q_1 = x$ and $P_1 = y$, it can be seen that

$$F(x) = Q_2 + \frac{ah(x)+b}{cah'(x)} f(x) \tag{3.2.4}$$

Therefore, replacing $F(x)$ in (3.2.3) by expression as given in (3.2.4), and noting (Ali and Khan, 1997) that

$$E[h(X_{r:n})] - E[h(X_{r-1:n})] \\ = \binom{n}{r-1}\int_{Q_1}^{P_1} h'(x)[F(x)]^{r-1}[1-F(x)]^{n-r+1}\, dx \tag{3.2.5}$$

we have,

$$E[h(X_{1:m})] - E[h(X_{1:m-1})] = -Q_2\{E[h(X_{1:m-1})] - h(Q_1)\} \\ - \frac{1}{mc}E[h(X_{1:m})] - \frac{b}{mca}$$

That is,

$$E[h(X_{1:m})]=-\frac{mc}{(mc+1)}P_2E[h(X_{1:m-1})]+\frac{mc}{(mc+1)}Q_2h(x)$$
$$-\frac{b}{a}\frac{1}{(mc+1)} \tag{3.2.6}$$

But since from (3.1.1)

$$E[h(X_{r+1:n})|X_{r:n}=x,X_{s:n}=y]=E[h(X_{1:m})|\,x\le X_{1:m}\le y]$$
$$,m=s-r-1 \tag{3.2.7}$$

Therefore, writing (3.2.6) recursively after noting that $E[h(X_{1:0})]=h(y)$, we establish (3.2.1).

To prove (3.2.1) implies (3.2.2), we have

$$\frac{m}{[F(y)-F(x)]^m}\int_x^y h(t)[F(y)-F(t)]^{m-1}f(t)dt=$$
$$(-1)^m\left[\frac{1-F(y)}{F(y)-F(x)}\right]^m h(y)\prod_{i=1}^{m}\frac{ic}{(ic+1)}$$
$$+\left[\frac{1-F(x)}{F(y)-F(x)}\right]h(x)\sum_{i=0}^{m-1}(-1)^i\left[\frac{1-F(y)}{F(y)-F(x)}\right]^i\prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1}$$
$$-\frac{b}{a}\sum_{i=0}^{m-1}(-1)^i\left[\frac{1-F(y)}{F(y)-F(x)}\right]^i\frac{1}{(m-i)c}\prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1}$$

Differentiating both the sides w.r.t. x, we get

$$-f(x)\Bigg[h(x)\Bigg\{mc[F(y)-F(x)]^{m-1}-c\sum_{i=0}^{m-1}(-1)^i\prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1}$$
$$[1-F(y)]^i[F(y)-F(x)]^{m-i-1}-[1-F(x)]c$$
$$\sum_{i=0}^{m-1}(-1)^i\prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1}(m-i-1)[1-F(y)]^i[F(y)-F(x)]^{m-i-2}\Bigg\}$$
$$+\frac{b}{a}\sum_{i=0}^{m-1}(-1)^i\prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1}[1-F(y)]^i[F(y)-F(x)]^{m-i-1}\Bigg]$$

$$= [1 - F(x)]ch'(x)$$

$$\left\{ \sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1} [1 - F(y)]^i [F(y) - F(x)]^{m-i-1} \right\} \tag{3.2.8}$$

Now consider,

$$mc[F(y) - F(x)]^{m-1}$$

$$- c \sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1} [1 - F(y)]^i [F(y) - F(x)]^{m-i-1}$$

$$- [1 - F(x)]c \sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1} (m-i-1)$$

$$[1 - F(y)]^i [F(y) - F(x)]^{m-i-2}$$

In the above expression write $[1 - F(x)]$ as $[\{F(y) - F(x)\} + \{1 - F(y)\}]$, which on solving equates to

$$\sum_{i=0}^{m-1} (-1)^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1} [1 - F(y)]^i [F(y) - F(x)]^{m-i-1}$$

and hence from (3.2.8) we have,

$$- f(x)\left[h(x) + \frac{b}{a}\right] = [1 - F(x)]ch'(x)$$

$$- \frac{f(x)}{1 - F(x)} = \frac{cah'(x)}{ah(x) + b}$$

Therefore $F(x) = 1 - [ah(x) + b]^c$

Examples on Theorem 3.2.1:

Proper choice of a, b, and $h(x)$ characterize the distributions as given in the table.

Distribution	$F(x)$	a	b	c	$h(x)$
1. Power function	$a^{-p}x^{p}, 0 \le x \le a$	$-a^{-p}$	1	1	x^{p}
2. Pareto	$1-a^{p}x^{-p}$, $a \le x \le \infty$	a^{p}	0	1	x^{-p}
		a	0	p	x^{-1}
		a^{-p}	0	-1	x^{p}
		a^{-1}	0	$-p$	x
3. Beta of the first kind	$1-(1-x)^{p}, 0 \le x \le 1$	-1	1	p	x
		1	0	p	$1-x$
		1	0	1	$(1-x)^{p}$
4. Weibull	$1-e^{-\theta x^{p}}, 0 \le x < \infty$	1	0	1	$e^{-\theta x^{p}}$
		1	0	θ	$e^{-x^{p}}$
5. Inverse Weibull	$e^{-\theta x^{-p}}, 0 \le x < \infty$	-1	1	1	$e^{-\theta x^{-p}}$
6. Burr type II	$(1+e^{-x})^{-k}$, $-\infty < x < \infty$	-1	1	1	$(1+e^{-x})^{-k}$
7. Burr type III	$(1+x^{-c})^{-k}, 0 \le x < \infty$	-1	1	1	$(1+x^{-c})^{-k}$
8. Burr type IV	$\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$, $0 \le k \le c$	-1	1	1	$\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$
9. Burr type V	$[1+ce^{-\tan x}]^{-k}$, $-\pi/2 \le x \le \pi/2$	-1	1	1	$[1+ce^{-\tan x}]^{-k}$
10. Burr type VI	$[1+ce^{-k\sinh x}]^{-k}$, $-\infty < x < \infty$	-1	1	1	$[1+ce^{-k\sinh x}]^{-k}$
11. Burr type VII	$2^{-k}(1+\tanh x)^{k}$, $-\infty < x < \infty$	-2^{-k}	1	1	$(1+\tanh x)^{k}$
12. Burr type VIII	$\left(\frac{2}{\pi}\tan^{-1} e^{x}\right)^{k}$, $-\infty < x < \infty$	$-\left(\frac{2}{\pi}\right)^{k}$	1	1	$(\tan^{-1} e^{x})^{k}$
13. Burr type IX	$1-\frac{2}{c[(1+e^{x})^{k}-1]+2}$ $-\infty < x < \infty$	$\frac{c}{2}$	$1-\frac{c}{2}$	-1	$(1+e^{x})^{k}$
14. Burr type X	$(1-e^{-x^{2}})^{k}$, $0 \le x < \infty$	-1	1	1	$(1-e^{-x^{2}})^{k}$

15. Burr type XI	$\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^k$, $0\le x\le 1$	-1	1	1	$\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^k$
16. Burr type XII	$1-(1+\theta x^p)^{-m}$, $0\le x<\infty$	θ	1	$-m$	x^p
17. Cauchy	$\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x$, $-\infty<x<\infty$	$-\frac{1}{\pi}$	$\frac{1}{2}$	1	$\tan^{-1}x$

Corollary 3.2.1: For any continuous and differentiable function $h()$ and $1\le r<j<s\le n$;

$$E[h(X_{j:n})|\,X_{r:n}=x, X_{s:n}=y]$$

$$=E[h(X_{j-r:s-r-1}),\, x\le X_{j-r:s-r-1}\le y]$$

$$=(s-j)\binom{s-r-1}{j-r-1}\sum_{l=0}^{j-r-1}(-1)^l\binom{j-r-1}{l}$$
$$\frac{1}{(s-j+l)}E[h(X_{1:s-j+l}),\, x\le X_{1:s-j+l}\le y]$$

Proof: In view of equation (1.5.5), we have

$$E[h(X_{j:n})|\,X_{r:n}=x, X_{s:n}=y]$$

$$=\frac{(s-r-1)!}{(j-r-1)!(s-j-1)!}$$
$$\int_x^y h(t)\frac{[F(t)-F(x)]^{j-r-1}[F(y)-F(t)]^{s-j-1}}{[F(y)-F(x)]^{s-r-1}}f(t)dt$$

Now writing the term $[F(t)-F(x)]^{j-r-1}$ in the integrand as $[\{F(y)-F(x)\}-\{F(y)-F(t)\}]^{j-r-1}$ and then expand it binomially, we get

$$E[h(X_{j:n})\mid X_{r:n}=x, X_{s:n}=y]$$

$$=\frac{(s-r-1)!}{(j-r-1)!(s-j-1)!}\sum_{l=0}^{j-r-1}(-1)^{l}\binom{j-r-1}{l}$$

$$\int_{x}^{y}h(t)\frac{[F(y)-F(t)]^{s-j+l-1}}{[F(y)-F(x)]^{s-j+l}}f(t)dt$$

and hence the result

Remark 3.2.1: At $s=n+1, X_{n+1:n}=y=\beta, P=1$

Therefore we have $P_2=0, Q_2=1$, $m=n-r$ and

$$E\left[h(X_{r+1:n})\middle|X_{r:n}=x\right]=0+h(x)\frac{mc}{(mc+1)}-\frac{b}{a}\frac{1}{mc}\frac{mc}{(mc+1)}$$

$$=\frac{acmh(x)-b}{a(mc+1)}$$

as obtained by Khan and Abu-Salih (1989).

Remark 3.2.2: At $r=0$, we have $X_{0:n}=x=\alpha, Q=0$ and $m=s-1$

Therefore, if $F()$ is replaced by $1-F()$, then

$$Q_2=\frac{1-Q}{P-Q}=\frac{Q}{Q-P}=0$$

$$P_2=\frac{1-P}{P-Q}=\frac{P}{Q-P}=-1$$

Now replacing l by $(s-l), F()$ by $1-F()$, we get

$$E\left[h(X_{1:n})\mid X_{0:n}=\alpha, X_{s:n}=y\right]=E\left[h(X_{1:n})\mid X_{s:n}=y\right]$$

$$= h(y)\prod_{i=1}^{s-1}\frac{(s-i)c}{(s-i)c+1} + 0 - \frac{b}{a}\sum_{i=0}^{s-2}\frac{1}{[s-(m-i)]c}\prod_{j=0}^{i}\frac{[s-(m-j)]c}{[s-(m-j)]c+1}$$

$$= h(y)\prod_{i=0}^{s-2}\frac{(s-1-i)}{(s-1-i)c+1} - \frac{b}{a}\sum_{i=0}^{s-2}\frac{1}{(i+1)c}\prod_{j=0}^{i}\frac{(j+1)c}{(j+1)c+1}$$

if and only if

$$F(x) = 1-[1-(ah(x)+b)^c] = [ah(x)+b]^c$$

This characterization result was given by Khan and Abouammoh (2000, Theorem 2.2, $r=1$). Power function distribution $\left(a = a^{-p}, b=0, c=1, h(x)=x^p\right)$ was characterized by Khan and Ali (1987).

Further, at $s=2$, we have

$$E[h(X_{1:n}) \mid X_{2:n}=y] = E[h(X) \mid X \le y] = \frac{ach(y)-b}{a(c+1)}$$

if and only if $F(x)=[ah(x)+b]^c$ [Khan and Abu-Salih, 1989].

Theorem 3.2.2: Under the conditions given in Theorem 2.1, and $1 \le r < s \le n, m = s-r-1$

$$E[h(X_{s-1:n}) \mid X_{r:n}=x, X_{s:n}=y] =$$

$$(-1)^m Q_3^m h(x)\prod_{i=1}^{m}\frac{ic}{(ic+1)} + P_3 h(y)\sum_{i=0}^{m-1}(-1)^i Q_3^i \prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1}$$
$$-\frac{b}{a}\sum_{i=0}^{m-1}(-1)^i Q_3^i \frac{1}{(m-i)c}\prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1} \qquad (3.2.9)$$

if and only if

$$F(x) = [ah(x)+b]^c, \ \alpha \le x \le \beta \qquad (3.2.10)$$

where $Q_3 = \frac{Q}{P-Q}$, $P_3 = \frac{P}{P-Q}$

Proof: The Theorem can be proved on the lines of Theorem 3.2.1 or else by noting the fact that the conditional distribution of $X_{r+1:n}$ given $X_{r:n} = x$ and $X_{s:n} = y$ from $F()$ is the same as the conditional distribution of $X_{n-r:n}$ given $X_{n-s+1:n} = x$ and $X_{n-r+1:n} = y$ from $1-F()$. Therefore, replacing $1-P$ by Q, $1-Q$ by P, x by y and y by x in Theorem 3.2.1, we have

$$E\left[h(X_{n-r:n}) \mid X_{n-s+1:n} = x, X_{n-r+1:n} = y\right]$$

$$= (-1)^m Q_3^m h(x) \prod_{i=1}^{m} \frac{ic}{(ic+1)} + P_3 h(y) \sum_{i=0}^{m-1} (-1)^i Q_3^i \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1}$$

$$- \frac{b}{a} \sum_{i=0}^{m-1} (-1)^i Q_3^i \frac{1}{(m-i)c} \prod_{j=0}^{i} \frac{(m-j)c}{(m-j)c+1}$$

if and only if

$$F(x) = 1 - [1 - (ah(x)+b)^c] = [ah(x)+b]^c$$

Now replacing $(n-s+1)$ by r and $(n-r+1)$ by s, the Theorem is proved.

Examples on Theorem 3.2.2:

Proper choice of a, b, and $h(x)$ characterize the distributions as given in the table.

Distribution	a	b	c	$h(x)$
1. Power function	a^{-p}	0	1	x^p
	a^{-1}	0	p	x
	a^p	0	-1	x^{-p}
	a	0	$-p$	x^{-1}
2. Pareto	$-a^p$	1	1	x^{-p}

3. Beta of the first kind	-1	1	1	$(1-x)^p$
4. Weibull	-1	1	1	$e^{-\theta x^p}$
5. Inverse Weibull	1	0	1	$e^{-\theta x^{-p}}$
	1	0	θ	$e^{-x^{-p}}$
6. Burr type II	1	1	$-k$	e^{-x}
7. Burr type III	1	1	$-k$	x^{-c}
8. Burr type IV	1	1	$-k$	$\left(\frac{c-x}{x}\right)^{1/c}$
9. Burr type V	c	1	$-k$	$e^{-\tan x}$
10. Burr type VI	c	1	$-k$	$e^{-k \sinh x}$
11. Burr type VII	$\frac{1}{2}$	$\frac{1}{2}$	k	$\tanh x$
12. Burr type VIII	$\frac{2}{\pi}$	0	k	$\tan^{-1} e^x$
13. Burr type IX	-1	1	1	$[c\{(1+e^x)^k - 1\} + 2]^{-1}$
14. Burr type X	-1	1	k	e^{-x^2}
15. Burr type XI	1	0	k	$\left(x - \frac{1}{2\pi}\sin 2\pi x\right)$
16. Burr type XII	-1	1	1	$(1+\theta x^p)^{-m}$
17. Cauchy	$\frac{1}{\pi}$	$\frac{1}{2}$	1	$\tan^{-1} x$

Corollary 3.2.2: For any continuous and differentiable function $h()$ and $1 \le r < j < s \le n$;

$$E[h(X_{j:n}) | X_{r:n} = x, X_{s:n} = y]$$

$$= E[h(X_{j-r:s-r-1}), x \le X_{j-r:s-r-1} \le y]$$

$$= (s-j)\binom{s-r-1}{j-r-1} \sum_{l=0}^{s-j-1} (-1)^l \binom{s-j-1}{l} \frac{1}{(j-r+l)} E[h(X_{j-r+l:j-r+l}), x \le X_{j-r+l:j-r+l} \le y]$$

Proof: This may be proved on the lines on Corollary 3.2.1, after expressing the term $[F(y)-F(t)]^{s-j-1}$ as $[\{F(y)-F(x)\}-\{F(t)-F(x)\}]^{s-j-1}$.

Remark 3.2.3: At $r=0$, we have $X_{0:n}=x=\alpha, Q=0, Q_3=0, P_3=1$ and $m=s-1$

Therefore,

$$E[h(X_{s-1:n})|\,X_{s:n}=y]=h(y)\frac{mc}{(mc+1)}-\frac{b}{a}\frac{1}{(mc+1)}$$

and $$E[h(X_{r:n})|\,X_{r+1:n}=y]=\frac{arch(y)-b}{a(rc+1)}$$

as obtained by Khan and Abu-Salih (1989).

Remark 3.2.4: At $s=n+1$, we have $X_{n+1:n}=y=\beta, P=1$ and $m=n-r$. Now if $F()$ is replaced by $1-F()$, then

$$Q_3=\frac{Q}{P-Q}=\frac{1-Q}{Q-P}=-1$$

$$P_3=\frac{P}{P-Q}=\frac{1-P}{Q-P}=0$$

Therefore proceeding as in Remark 3.2.2, we get

$$E[h(X_{n:n})|\,X_{r:n}=x]$$
$$=h(x)\prod_{i=0}^{n-r-1}\frac{(n-r-i)c}{(n-r-i)c+1}-\frac{b}{a}\sum_{i=0}^{n-r-1}\frac{1}{(i+1)}\prod_{j=0}^{i}\frac{(j+1)c}{(j+1)c+1}$$

if and only if $F(x)=1-[ah(x)+b]^c$

This result was given by Khan and Abouammoh (2000, Theorem 2.1) for $s=n$.

At $r=n-1$,

$$E[h(X_{n:n})|\,X_{n-1:n}=x]=E[h(X)|\,X\geq x]$$

$$=\frac{ach(x)-b}{a(c+1)}$$

as given by Khan and Abu-Salih (1989).

Theorem 3.2.3: Under the conditions given in Theorem 2.1, and for $1\leq r<s\leq n, m=s-r-1$,

$$E[h(X_{r+1:n})|\,X_{r:n}=x,X_{s:n}=y]$$
$$=(-1)^m P_2^m h(y)+Q_2 h(x)\sum_{i=0}^{m-1}(-1)^i P_2^i+\frac{1}{a}\sum_{i=0}^{m-1}(-1)^i\frac{1}{(m-i)}P_2^i \tag{3.2.11}$$

if and only if

$$F(x)=1-be^{-ah(x)},\ \alpha\leq x\leq\beta \tag{3.2.12}$$

for $a\neq 0,(m-i)\neq 0, be^{ah(\alpha)}=1,$

Proof: To prove necessity, note that

$$F(x)=Q_2-\frac{f(x)}{ah'(x)}$$

Using the relations (3.2.3) and (3.2.5), it can be shown that

$$E[h(X_{1:m})]=-P_2E[h(X_{1:m-1})]+Q_2h(x)+\frac{1}{ma}$$

which on writing recursively and noting the relation given in (3.2.7), one can establish (3.2.11).

To prove sufficiency, proceed on the lines of Theorem 3.2.1 and differentiate both sides w.r.t. x, to get

$$
\begin{aligned}
&-mh(x)[F(y)-F(x)]^{m-1}f(x)\\
&=0+[1-F(x)]h'(x)\sum_{i=0}^{m-1}(-1)^i[1-F(y)]^i[F(y)-F(x)]^{m-i-1}\\
&-f(x)h(x)\sum_{i=0}^{m-1}(-1)^i[1-F(y)]^i\,[F(y)-F(x)]^{m-i-1}\\
&-[1-F(x)]h(x)f(x)\sum_{i=0}^{m-1}(-1)^i\,(m-i-1)[1-F(y)]^i\\
&[F(y)-F(x)]^{m-i-2}-\frac{f(x)}{a}\sum_{i=0}^{m-1}(-1)^i[1-F(y)]^i[F(y)-F(x)]^{m-i-1}\\
&\frac{f(x)}{a}\Bigg\{-mah(x)[F(y)-F(x)]^{m-1}+ah(x)\sum_{i=0}^{m-1}(-1)^i[1-F(y)]^i\\
&\qquad[F(y)-F(x)]^{m-i-1}+ah(x)[1-F(x)]\sum_{i=0}^{m-1}(-1)^i(m-i-1)\\
&\qquad[1-F(y)]^i[F(y)-F(x)]^{m-i-2}\\
&\qquad+\sum_{i=0}^{m-1}(-1)^i[1-F(y)]^i[F(y)-F(x)]^{m-i-1}\Bigg\}\\
&\qquad=h'(x)[1-F(x)]\left\{\sum_{i=0}^{m-1}(-1)^i[1-F(y)]^i[F(y)-F(x)]^{m-i-1}\right\}
\end{aligned}
\tag{3.2.13}
$$

In the above equation consider the term

$$
\begin{aligned}
&-mah(x)[F(y)-F(x)]^{m-1}\\
&\;+ah(x)\sum_{i=0}^{m-1}(-1)^i[1-F(y)]^i[F(y)-F(x)]^{m-i-1}+ah(x)[1-F(x)]\\
&\qquad\qquad\sum_{i=0}^{m-1}(-1)^i(m-i-1)[1-F(y)]^i[F(y)-F(x)]^{m-i-2}
\end{aligned}
$$

and express the $[1-F(x)]$ as $[\{F(y)-F(x)\}+\{1-F(y)\}]$, which on solving equates to zero.

Hence from (3.2.13), we have

$$-\frac{f(x)}{1-F(x)}=-ah'(x)$$

giving

$$F(x)=1-be^{-ah(x)}$$

Examples on Theorem 3.2.3:

Distribution	a	b	$h(x)$
1. Pareto	p	$p\ln a$	$\ln x$
2. Beta of the first Kind	$-p$	0	$\ln(1-x)$
3. Weibull	θ	0	x^p
4. Burr type XII	m	0	$\ln(1+\theta x^p)$

Remark 3.2.5: At $s=n+1$, it reduces to the Theorem 2.2 of Khan and Abu- Salih (1989) with b replaced by e^b.

Remark 3.2.6: At $r=0$, proceeding on the lines of Remark 2.2, it can be shown that

$$E[h(X_{1:n})\mid X_{s:n}=y]=h(y)+\frac{1}{a}\sum_{j=1}^{s-1}\frac{1}{j}$$

as obtained by Khan and Abouammoh (2000) for $r=1$.

For $s=2$,

$$E[h(X_{1:n})\mid X_{2:n}=y]=E[h(X)\mid X\le y]$$

$$=h(y)+\frac{1}{a}$$

if and only if $F(x) = be^{-ah(x)}$

[Khan and Abu-Salih (1989), Ouyang (1995)] and Editorial notes $[Sankhy\bar{a}, Ser.A, 60(1998), \mathrm{p}150]$.

Theorem 3.2.4: Under the conditions given in Theorem 3.2.1 and for $1 \le r < s \le n$,

$$E[h(X_{s-1:n}) | X_{r:n} = x, X_{s:n} = y]$$
$$= (-1)^m Q_3^m h(x) + P_3 h(y) \sum_{i=0}^{m-1} (-1)^i Q_3^i + \frac{1}{a} \sum_{i=0}^{m-1} (-1)^i \frac{1}{(m-i)} Q_3^i$$

if and only if

$$F(x) = be^{-ah(x)}$$

for $a \ne 0, m - i \ne 0, b = e^{ah(\beta)}, m = s - r - 1$

Proof: To prove the Theorem, proceed on the lines of Theorem 3.2.2. Also please note that at r=0, it reduces to Lemma 2.2 of Khan and Abu-Salih (1989) with b replaced by e^b. Further, at $s = n + 1$

$$E[h(X_{n:n}) | X_{r:n} = x] = h(x) + \frac{1}{a} \sum_{j=r}^{n-1} \frac{1}{(n-j)}$$

as given by Khan and Abouammoh (2000) and Wu and Ouyang (1996).

Further, at $r = n - 1$,

$$E[h(X_{n:n}) | X_{n-1:n} = x] = E[h(X) | X \ge x]$$

$$= h(x) + \frac{1}{a}$$

if and only if $F(x) = 1 - be^{-ah(x)}$

Examples on Theorem 3.2.4:

Distribution	a	b	$h(x)$
1. Power function	$-p$	$\ln a^{-p}$	$\ln x$
2. Inverse Weibull	θ	0	x^{-p}
3. Burr type II	k	0	$\ln(1+e^{-x})$
4. Burr type III	k	0	$\ln(1+x^{-c})$
5. Burr type IV	k	0	$\ln\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]$
6. Burr type V	k	0	$\ln(1+ce^{-\tan x})$
7. Burr type VI	k	0	$\ln(1+ce^{-\sinh x})$
8. Burr type VII	$-k$	0	$\ln\left(\frac{1+\tanh x}{2}\right)$
9. Burr type VIII	$-k$	0	$\ln\left(\frac{2}{\pi}\tan^{-1} e^{x}\right)$
10. Burr type X	$-k$	0	$\ln(1+e^{-x^2})$
11. Burr type XI	$-k$	0	$\ln\left(x-\frac{1}{2\pi}\sin 2x\right)$
12. Cauchy	-1	0	$\ln\left(\frac{1}{2}+\frac{1}{\pi}\tan^{-1} x\right)$

Chapter IV

Chapter IV

CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH CONDITIONAL EXPECTATION OF FUNCTION OF TWO ORDER STATISTICS

4.1 Introduction

Here in this Chapter a class of distribution $F(x) = ah(x) + b$ has been characterized by considering conditional moments of function of two order statistics, conditioned on pair of order statistics. Further, its important deductions are also discussed.

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a continuous population having probability density function $(pdf)\, f(x)$ and distribution function $(df)\, F(x)$, over the support (α, β) and let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the corresponding order statistics. Balasubramanian and Beg (1992) characterized some distributions through the relation

$$E[h(X) \mid x \leq X \leq y] = \frac{h(x) + h(y)}{2} \tag{4.1.1}$$

where $h(x)$ is a measurable function of x.

Khan and Athar (2000b) extended the result of Balasubramanian and Beg (1992) to order statistics and characterized a general class of distribution $F(x) = [ah(x) + b]$ using the relation

$$E[h(X_{r+i:n})|\,X_{r:n}=x,X_{s:n}=y]=\frac{(m-i+1)h(x)+ih(y)}{(m+1)} \quad (4.1.2)$$

However, for the expression

$$E[h(X_{j:n}).h(X_{k:n})|\,X_{r:n}=x,X_{s:n}=y] \quad (4.1.3)$$

perhaps no such results are available in the literature. We have, therefore made an attempt to characterize a family of distribution utilizing conditional expectation (4.1.3).

The doubly truncated *pdf* of continuous random variable is denoted as

$$f^*(x)=\frac{f(x)}{P-Q},\ Q_1<x<P_1 \quad (4.1.4)$$

where $F(Q_1)=Q,\ F(P_1)=P$

and the *df* is

$$F^*(x)=\frac{F(x)-Q}{P-Q} \quad (4.1.5)$$

The *pdf* of the $X_{r:n}$, $1\le r\le n$, is given by

$$f_{r:n}(x)=C_{r:n}[F^*(x)]^{r-1}[1-F^*(x)]^{n-r}f^*(x) \quad (4.1.6)$$

and the joint *pdf* of $X_{r:n}$ and $X_{s:n}$, $1\le r<s\le n$, is

$$f_{r,s:n}(x)=C_{r,s:n}[F^*(x)]^{r-1}[F^*(y)-F^*(x)]^{s-r-1}$$
$$[1-F^*(y)]^{n-s}f^*(x)f^*(y) \quad (4.1.7)$$

where $C_{r:n} = \dfrac{n!}{(r-1)!(n-r)!}$

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

Also, we shall use the convention

$$X_{0:n} = Q_1 \text{ and } X_{n:n-1} = P_1, \quad n = 1,2,3,\ldots$$

4.2 Characterization Theorem

Before coming to the main result we shall prove the following lemmas:

Lemma 4.2.1: Conditional distribution of $(X_{j:n}, X_{k:n})$ given $X_{r:n} = x$ and $X_{s:n} = y$, $1 \le r < j < k < s \le n$, is unconditional distribution of $(X_{j-r:s-r-1}, X_{k-r:s-r-1})$ truncated to the left at x and to the right at y.

Proof: This can be proved by writing the joint *pdf* of $X_{j:n}$, $X_{k:n}$, $X_{r:n}$ and $X_{s:n}$ in the numerator and the joint *pdf* of $X_{r:n}$ and $X_{s:n}$ in the denominator and then simplifying.

Lemma 4.2.2: For $1 \le r < s \le n$, $n = 1,2,\ldots$

$$E[g(X_{r:n}, X_{s:n})] - E[g(X_{r-1:n}, X_{s:n})]$$

$$= \frac{C_{r,s:n}}{(s-r)} \int_{Q_1}^{P_1} \int_x^{P_1} \frac{\partial}{\partial x} g(x,y)[F^*(x)]^{r-1}[F^*(y) - F^*(x)]^{s-r}$$

$$[1 - F^*(y)]^{n-s} f^*(y) dy dx \quad (4.2.1)$$

where $g(x,y)$ is a measurable function of (x,y).

Proof: We have,

$$E[g(X_{r:n},X_{s:n})]-E[g(X_{r-1:n},X_{s:n})]$$

$$=\frac{C_{r,s:n}}{(s-r)}\int_{Q_1}^{P_1}\int_{x}^{P_1} h(x)h(y)[F^*(x)]^{r-2}[F^*(y)-F^*(x)]^{s-r-1}$$

$$[1-F^*(y)]^{n-s}\{(s-r)F^*(x)-(r-1)[F^*(y)-F^*(x)]\}$$

$$f^*(x)f^*(y)dxdy$$

let $\quad k(x,y)=-[F^*(x)]^{r-1}[F^*(y)-F^*(x)]^{s-r}$

$$\frac{\partial}{\partial x}k(x,y)=(s-r)[F^*(x)]^{r-1}[F^*(y)-F^*(x)]^{s-r-1}f^*(x)$$

$$-(r-1)[F^*(x)]^{r-2}[F^*(y)-F^*(x)]^{s-r}f^*(x)$$

$$=[F^*(x)]^{r-2}[F^*(y)-F^*(x)]^{s-r-1}$$

$$\{(s-r)F^*(x)-(r-1)[F^*(y)-F^*(x)]\}f^*(x)$$

Then,

$$E[g(X_{r:n},X_{s:n})]-E[g(X_{r-1:n},X_{s:n})]$$

$$=\frac{C_{r,s:n}}{(s-r)}\int_{Q_1}^{P_1}\int_{x}^{P_1} h(x)h(y)[1-F^*(y)]^{n-s}\left(\frac{\partial}{\partial x}k(x,y)\right)f^*(y)dxdy$$

$$=\frac{C_{r,s:n}}{(s-r)}\int_{Q_1}^{P_1} h(y)[1-F^*(y)]^{n-s}f^*(y)\left\{\int_{Q_1}^{y} h(x)\frac{\partial}{\partial x}k(x,y)dx\right\}dy$$

$$=\frac{C_{r,s:n}}{(s-r)}\int_{Q_1}^{P_1}\int_{x}^{P_1} h'(x)h(y)[F^*(x)]^{r-1}[F^*(y)-F^*(x)]^{s-r}$$

$$[1-F^*(y)]^{n-s}f^*(y)dxdy$$

$$=\frac{C_{r,s:n}}{(s-r)}\int_{Q_1}^{P_1}\int_{x}^{P_1}\frac{\partial}{\partial x}g(x,y)[F^*(x)]^{r-1}[F^*(y)-F^*(x)]^{s-r}$$

$$[1-F^*(y)]^{n-s}f^*(y)dxdy$$

and hence the result.

Lemma 4.2.3: For $1 \le r < s \le n$, $n = 1, 2, \ldots$ and $F(x) = ah(x) + b$, where $a \neq 0, b, c \neq 0$ are finite constants and $h(x)$ is differentiable function of x in the interval (α, β).

$$E[g(X_{r:n}, X_{s:n}) = E[g(X_{r-1:n}, X_{s:n})] + \frac{(P-Q)}{(n+1)a} E[h(X_{s+1:n+1})] \tag{4.2.2}$$

where $g(x, y) = h(x).h(y)$

Proof: For $F(x) = ah(x) + b$, we have

$$1 = \frac{(P-Q)}{ah'(x)} f^*(x) \tag{4.2.3}$$

Using lemma (4.2.2), we have

$$E[g(X_{r:n}, X_{s:n})] - E[g(X_{r-1:n}, X_{s:n})]$$

$$= \frac{C_{r,s:n}}{(s-r)} \int_{Q_1}^{P_1} \int_{x}^{P_1} h'(x) h(y) [F^*(x)]^{r-1} [F^*(y) - F^*(x)]^{s-r}$$

$$[1 - F^*(y)]^{n-s} \left\{ \frac{(P-Q)}{ah'(x)} f^*(x) \right\} f^*(y)\, dy\, dx$$

$$= \frac{(P-Q)}{(n+1)a} E[h^0(X_{r:n+1}) h(X_{s+1:n+1})]$$

and hence the result.

Theorem 4.2.1: For any continuous and differentiable function $h()$ and $m = s - r - 1, 1 \le r < s \le n, p < q$ with $g(x, y) = h(x).h(y)$.

$$E[g(X_{r+p:n}, X_{r+q:n}) | X_{r:n} = x, X_{s:n} = y]$$

$$= \frac{1}{(m+1)(m+2)}[(m - p + 2)(m - q + 1)h^2(x) + p(q + 1)h^2(y) \\ + \{(p + q)m - 2q(p - 1)\}h(x)h(y)] \quad (4.2.4)$$

if and only if

$$F(x) = ah(x) + b, \ \alpha \le x \le \beta \quad (4.2.5)$$

with $F(\alpha) = 0$ and $F(\beta) = 1$.

Proof: For the necessary part, we have,

$$E[g(X_{r+p:n}, X_{r+q:n}) | X_{r:n} = x, X_{s:n} = y] = E[g(X_{p:m}, X_{q:m})], \\ x \le X_{p:m} < X_{q:m} \le y \quad (4.2.6)$$

from Lemma 4.2.1.

But from Lemma 4.2.3 for $F(x) = ah(x) + b$,

$$E[g(X_{p:m}, X_{q:m})] \\ = E[g(X_{p-1:m}, X_{q:m}) + \frac{(P - Q)}{(m+1)a} E[h(X_{q+1:m+1})] \quad (4.2.7)$$

where $F(x) = Q$ and $F(y) = P$

Since $E[h(X_{0:m})] = h(x)$. Therefore, for $g(x, y) = h(x).h(y)$ at $p = 1$, we have

$$E[h(X_{1:m}).h(X_{q:m})]$$
$$= h(x)E[h(X_{q:m})] + \frac{[h(y)-h(x)]}{(m+1)} E[h(X_{q+1:m+1})] \qquad (4.2.8)$$

Writing recursively, we have

$$E[h(X_{p:m}).h(X_{q:m})]$$
$$= h(x)E[h(X_{q:m})] + \frac{p[h(y)-h(x)]}{(m+1)} E[h(X_{q+1:m+1})] \qquad (4.2.9)$$

Also since (Ali and Khan, 1997)

$$E[h(X_{r:m})] - E[h(X_{r-1:m})] = \frac{[h(y)-h(x)]}{(m+1)}$$

Therefore,

$$E[h(X_{r:m})] = h(x) + \frac{r[h(y)-h(x)]}{(m+1)} \qquad (4.2.10)$$

Hence in view of (4.2.6), (4.2.9) and (4.2.10), the relation (4.2.4) is established.

To prove sufficiency part we have, from the lemma (4.2.1), (4.2.2) and (4.2.3),

$$E[g(X_{p:m}, X_{q:m})] - E[g(X_{p-1:m}, X_{q:m})$$
$$= \frac{F(y)-F(x)}{(m+1)a} E[h(X_{q+1:m+1})]$$

That is,

$$\frac{m!}{(p-1)!(q-p)!(m-q)!} \frac{1}{[F(y)-F(x)]^m}$$
$$\int_x^y \int_x^{t_2} h'(t_1)h(t_2)[F(t_1)-F(x)]^{p-1}[F(t_2)-F(t_1)]^{q-p}$$
$$[F(y)-F(t_2)]^{m-q} f(t_2)\,dt_1\,dt_2$$

$$=\frac{(m+1)![F(y)-F(x)]}{q!(m-q)!(m+1)a}\frac{1}{[F(y)-F(x)]^{m+1}}\int_x^y h(t)[F(t)-F(x)]^q [F(y)-F(t)]^{m-q} f(t)dt$$

$$\frac{a}{(p-1)!(q-p)!}\int_x^y h(t_2)[F(y)-F(t_2)]^{m-q} f(t_2) \left\{\int_x^{t_2} h'(t_1)[F(t_2)-F(t_1)]^{q-p}[F(t_1)-F(x)]^{p-1}dt_1\right\}dt_2$$

$$=\frac{1}{q!}\int_x^y h(t)[F(t)-F(x)]^q[F(y)-F(t)]^{m-q} f(t)dt$$

Differentiating both sides $(m-q+1)$ times w.r.t. y, we get

$$\frac{a}{(p-1)!(q-p)!}\int_x^y h'(t_1)[F(t_1)-F(x)]^{p-1}[F(y)-F(t_1)]^{q-p}dt_1 =\frac{1}{q!}[F(y)-F(x)]^q$$

Differentiating again both sides once w.r.t. y, we get

$$\frac{a}{(p-1)!(q-p)!}(q-p)\int_x^y h'(t_1)[F(t_1)-F(x)]^{p-1} [F(y)-F(t_1)]^{q-p-1}dt_1 =\frac{1}{q!}q[F(y)-F(x)]^{q-1}$$

Differentiating again w.r.t. y, we have

$$\frac{a}{(p-1)!(q-p)!}(q-p)(q-p-1)\int_x^y h'(t_1)[F(t_1)-F(x)]^{p-1} [F(y)-F(t_1)]^{q-p-2}dt_1 =\frac{1}{q!}q(q-1)[F(y)-F(x)]^{q-1}$$

.

.

.

Differentiation at the $(q-p+1)th$ times gives,

$$\frac{a}{(p-1)!(q-p)!}(q-p)!h'(y)[F(y)-F(x)]^{p-1}$$

$$=\frac{q(q-1)(q-2).....p}{q!}[F(y)-F(x)]^{p-1}f(y)$$

$\Rightarrow ah'(y)=f(y)$,

$\Rightarrow F(y)=ah(y)+b$

where a and b satisfy the initial condition of *df* $F(y)$.

This proves the theorem.

Remark 4.2.1: At $p=1$, $q=2$ and $s=r+2, m=s-r-1=1$

$$E[h(X_{r+1:n}).h(X_{r+2:n})| X_{r:n}=x, X_{r+2:n}=y]$$

$$=h(y)\,E[h(X_{r+1:n})|X_{r:n}=x, X_{r+2:n}=y]$$

$$=h(y)\,E[h(X)|\,x\le X\le y]$$

$$=h(y)\frac{h(x)+h(y)}{2} \tag{4.2.11}$$

as obtained by Balasubramanian and Beg (1992).

Remark 4.2.2: At $p=0$, after noting that $X_{0:n}=Q_1=x$

$$E[h(X_{r+q:n})|\,X_{r:n}=x, X_{s:n}=y]=\frac{(m-q+1)h(x)+qh(y)}{(m+1)} \tag{4.2.12}$$

Also, at $q=s-r$ and $X_{m+1:m}=P_1=y$

$$E[h(X_{r+p:n})|\,X_{r:n}=x, X_{s:n}=y]=\frac{(m-p+1)h(x)+ph(y)}{(m+1)} \tag{4.2.13}$$

as obtained by Khan and Athar (2000b).

Remark 4.2.3: At $s = n+1, X_{n+1:n} = y = \beta, m = n-r$,

$$E[h(X_{r+q:n}) | X_{r:n} = x] = \frac{(n-r-q+1)h(x) + qh(\beta)}{(n-r+1)} \tag{4.2.14}$$

as given by Franco and Ruiz (1997).

Further at $y = \beta$,

$$F(\beta) = ah(\beta) + b = 1 \ , \ h(\beta) = \frac{1-b}{a}$$

Hence (2.13) may be re-written as

$$E[h(X_{r+q:n}) | X_{r:n} = x] = \frac{(n-r-q+1)h(x)}{(n-r+1)} + \frac{q}{n-r+1} \frac{(1-b)}{a} \tag{4.2.15}$$

if and only if

$$F(x) = -ah(x) + (1-b)$$

$$= 1 - [ah(x) + b]$$

as obtained by Khan and Abouammoh (2000) at $c = 1$. At $q = 1$, the result was given by Khan and Abu-Salih(1989).

Remark 4.2.4: At $r = 0, X_{0:n} = x = \alpha$, $m = s-1$

$$E[h(X_{q:n}) | X_{s:n} = y] = \frac{(s-q)h(\alpha) + qh(y)}{s} \tag{4.2.16}$$

as given by Franco and Ruiz (1997).

Now since $F(\alpha) = ah(\alpha) + b = 0$

Hence, $E[h(X_{q:n}) \mid X_{s:n} = y] = \frac{q}{s} h(y) - \frac{(s-q)}{s} \frac{b}{a}$

as obtained by Khan and Abouammoh (2000) at $c = 1$.

Further, at $s = q + 1$, the result was derived by Khan and Abu-Salih (1989).

For the examples of the above distribution one may refer to Chapter II.

Chapter V

ON CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH CONDITIONAL EXPECTATION OF ORDER STATISTICS

5.1. Introduction

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a continuous population having probability density function $(pdf)\, f(x)$ and distribution function $(df)\, F(x)$, over the support (α, β) and let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the corresponding order statistics

Khan and Abu-Salih (1989) have characterized general form of distributions through conditional expectations of function of order statistics and established

i) $$E[h(X_{r+1:n}) | X_{r:n} = x] = \frac{ac(n-r)h(x) - b}{a[(n-r)c+1]}$$

if and only if

$$F(x) = 1 - [ah(x) + b]^c \tag{5.1.1}$$

and

ii) $$E[h(X_{r:n}) | X_{r+1:n} = x] = \frac{acrh(x) - b}{a[rc+1]}$$

if and only if

$$F(x) = [ah(x) + b]^c \tag{5.1.2}$$

Here it may be noted that expectation of $h(X)$, a function in $F(x)$ is considered. Further, most of the distributions considered by Khan and Abu-Salih (1989) are obtained at $c = 1$.

Therefore, we have made an attempt to characterize

$$F(x) = ah(x) + b$$

combining the two general form of distributions, through the conditional moments

$$E[g(X_{r+1:n}) | X_{r:n} = x] \text{ and } E[g(X_{r:n}) | X_{r+1:n} = x]$$

where $g(x) = e^{-h(x)}$, different from $h(x)$.

To this end, we note that (David, 1981)

The conditional distribution of $X_{s:n}$ given $X_{r:n} = x, 1 \le r < s \le n$ is unconditional distribution of $X_{s-r:n-r}$ truncated to the left at x.

That is,

$$E[h(X_{s:n}) | X_{r:n} = x] = E[h(X_{s-r:n-r}) | X_{s-r:n-r} \ge x] \tag{5.1.3}$$

and $$E[h(X_{n:n}) | X_{n-1:n} = x] = E[h(X) | X \ge x] \tag{5.1.4}$$

The conditional distribution of $X_{r:n}$ given $X_{s:n} = y$, $1 \le r < s \le n$ is unconditional distribution of $X_{r:s-1}$ truncated to the right at y.

That is,

$$E[h(X_{r:n})|\,X_{s:n}=y]=E[h(X_{r:s-1})|\,X_{r:s-1}\le y] \qquad (5.1.5)$$

and $$E[h(X_{1:n})|\,X_{2:n}=y]=E[h(Y)|\,Y\le y] \qquad (5.1.6)$$

The doubly truncated *pdf* of continuous random variable is denoted as

$$\frac{f(x)}{P-Q},\ Q_1<x<P_1 \qquad (5.1.7)$$

where $F(Q_1)=Q,\ F(P_1)=P$

and the *df* is

$$\frac{F(x)-Q}{P-Q} \qquad (5.1.8)$$

Also, we shall use the convention

$$X_{0:n}=Q_1 \text{ and } X_{n:n-1}=P_1,\quad n=1,2,3,...$$

5.2 Characterization Theorem

Theorem 2.1: For any continuous and differentiable function $g()=e^{-h()}$ and $m=n-r,\ 1\le r\le n$.

$$E[g(X_{r+1:n})|\,X_{r:n}=x]=$$

$$g(x)\sum_{i=1}^{m}(-1)^{i+1}\frac{m!}{(m-i)!}\frac{a^i}{[1-F(x)]^i}+g(\beta)(-1)^m m!\frac{a^m}{[1-F(x)]^m} \qquad (5.2.1)$$

if and only if

$$F(x)=ah(x)+b,\ \alpha\le x\le\beta \qquad (5.2.2)$$

with $F(\alpha)=0$, $F(\beta)=1$ and $g(\beta)=\exp\{(b-1)/a\}$

Proof: To prove (5.2.2) implies (5.2.1), we have for $F(x)=ah(x)+b$ and $g(x)=e^{-h(x)}$ (Ali and Khan, 1997),

$$E[g(X_{r:m})]=\frac{am}{(P-Q)}[E\{g(X_{r-1:m-1})\}-E\{g(X_{r:m-1})\}] \quad (5.2.3)$$

since $Z(x)=\dfrac{g'(x)}{h'(x)}=-h(x)$ at $c=1$

Writing (5.2.3) recursively after noting that

$$E[g(X_{r+1:n})| X_{r:n}=x]=E[g(X_{1:m})| X_{1:m}\geq x] \quad (5.2.4)$$

$$P=1, Q=F(x), E[g(X_{0:m-1})]=E[g(Q_1)]=g(x)$$

and $E[g(X_{1:0})]=E[g(\beta)]=g(\beta)$

the relation (5.2.1) is established.

To prove (5.2.1) implies (5.2.2), we have

$$\frac{m}{[1-F(x)]^m}\int_x^\beta g(t)[1-F(t)]^{m-1}f(t)dt$$
$$=g(x)\sum_{i=1}^{m}(-1)^{i+1}\frac{m!}{(m-i)!}\frac{a^i}{[1-F(x)]^i}+g(\beta)(-1)^m m!\frac{a^m}{[1-F(x)]^m}$$

or,

$$m\int_x^\beta g(t)[1-F(t)]^{m-1}f(t)dt$$
$$=g(x)\sum_{i=1}^{m}(-1)^{i+1}\frac{m!}{(m-i)!}a^i[1-F(x)]^{m-i}+g(\beta)(-1)^m m!a^m$$

Differentiating both the sides $w.r.t\ x$, we have

$$-mg(x)[1-F(x)]^{m-1}f(x)$$

$$=-g(x)\sum_{i=1}^{m}(-1)^{i+1}\frac{m!}{(m-i)!}a^{i}(m-i)[1-F(x)]^{m-i-1}f(x)$$

$$+g'(x)\sum_{i=1}^{m}(-1)^{i+1}\frac{m!}{(m-i)!}a^{i}[1-F(x)]^{m-i}$$

Dividing both sides by $-g(x)$ and after noting that $g'(x)=-h'(x).g(x)$, we get

$$m[1-F(x)]^{m-1}f(x)$$

$$=\sum_{i=1}^{m}(-1)^{i+1}\frac{m!}{(m-i)!}a^{i}(m-i)[1-F(x)]^{m-i-1}f(x)$$

$$+h'(x)\sum_{i=1}^{m}(-1)^{i+1}\frac{m!}{(m-i)!}a^{i}[1-F(x)]^{m-i}$$

$$\Rightarrow\quad ah'(x)\sum_{i=1}^{m}(-1)^{i+1}\frac{m!}{(m-i)!}a^{i-1}[1-F(x)]^{m-i}$$

$$=f(x)\sum_{i=1}^{m}(-1)^{i+1}\frac{m!}{(m-i)!}a^{i-1}[1-F(x)]^{m-i}$$

That is,

$$f(x)=a\,h'(x)$$

or, $\quad F(x)=a\,h(x)+b$

Hence the theorem.

Theorem 5.2.2: For any continuous and differentiable function $g()=e^{-h()}$ and $1\le r\le n$.

$$E[g(X_{r:n})| X_{r+1:n} = y] = g(\alpha)\frac{r!a^r}{[F(y)]^r} - g(y)\sum_{i=1}^{r}\frac{r!}{(r-i)!}\frac{a^i}{[F(y)]^i} \tag{5.2.5}$$

if and only if

$$F(x) = ah(x) + b, \ \alpha \le x \le \beta \tag{5.2.6}$$

with $F(\alpha) = 0,\ F(\beta) = 1$ and $g(\alpha) = e^{b/a}$.

Proof: The theorem can be proved on the lines of Theorem 5.2.1 after noting that

$$E[g(X_{r:n})| X_{r+1:n} = y] = E[g(X_{r:r})| X_{r:r} \le y]$$

$$P = F(y), Q = 0, E[g(X_{r:r-1})] = E[g(P_1)] = g(y)$$

and $E[g(X_{0:0})] = E[g(\alpha)] = g(\alpha)$

Remark 5.2.1: From Theorem 5.2.1 and equation (5.1.4), we have at $r = n-1$

$$E[g(X)| X \ge x] = \frac{a[g(x) - g(\beta)]}{[1 - F(x)]}$$

if and only if

$F(x) = ah(x) + b$, where $g(x) = e^{-h(x)}$.

Remark 5.2.2: From Theorem 5.2.2 and equation (5.1.6), we deduce at $r = 1$

$$E[g(Y)| Y \le y] = \frac{a[g(\alpha) - g(y)]}{F(y)}$$

if and only if

$$F(x) = ah(x) + b, \text{ where } g(y) = e^{-h(y)}.$$

For the examples of above distribution one may refer to Chapter II.

References

REFERENCES

Ali, M.A. and Khan, A.H. (1997): Recurrence relations for the expectations of a function of single order statistic from a general class of distributions. *J. Indian Statist. Assoc.*, 35, 1-9.

Ali, M.A. and Khan, A.H. (1998): Recurrence relations for expected values of certain functions of two order statistics. *Metron*, **46**, 107-119.

Arnold, B.C. and Balakrishnan, N. (1989): Relations, Bounds and Approximations for Order Statistics. *Lecture Notes in Statistics*, **Vol. 53**, Springer-Verlag, Berlin.

Arnold, B.C.; Balakrishnan, N. and Nagaraja, H.N. (1992): A First Course in Order Statistics. *John Wiley, New York.*

Balakrishnan, N. and Joshi, P.C. (1981): Moments of order statistics from doubly truncated power function distribution. *Aligarh Journal of Statistics*, **No.2**, 98-105.

Balakrishnan, N. and Malik, H.J. (1986): Order statistics from the linear exponential distribution, Part II: Increasing hazard rate case. *Commun. Statist. - Theor. Meth.*, **15(1)**, 179-203.

Balakrishnan, N; Malik, H.J. and Ahmed, S.E. (1988): Recurrence relations and identities for moments of order statistics, II: Specific continuous distributions. *Commun. Statist. - Theor. Meth.*, **17(8)**, 2657-2694.

Balakrishnan, N. and Cohen, A.C. (1991): Order Statistics and Inference: Estimation Methods. *Academic, Boston.*

Balasubramanian, K. and Beg, M.I. (1992): Distributions determined by conditioning on a pair of order statistics. *Metrika*, **39**, 107-112.

Balasubramanian, K. and Dey, A. (1997): Distributions characterized through conditional expectations. *Metrika*, **45**, 189-196.

Barnett, V. and Lewis, T. (1984): Outliers in Statistical Data. **2nd ed.**, *Wiley, New York.*

Beg, M.I. and Kirmani, S.N.U.A. (1978): Characterization of the exponential distributions by a weak homoscedasticity. *Comm. Statist. – Theor. Meth.*, **A7(3)**, 307-310.

Beg, M.I. and Balasubramanian, K. (1990): Distributions determined by conditioning on a single order statistic. *Metrika*, **37**, 37-43.

Blaquez, F.L. and Rebollo, J.L.M. (1997): A characterization of distributions based on linear regression of order statistics and record values. *Sankhyā*, *Ser.A*, **59**, 311-323.

David, H.A. (1981): Order statistics, *John Wiley and Sons, New York.*

Dimaki, C. and Xekalaki, E. (1996): Towards a unification of certain characterizations by conditional expectations. *Ann. Inst. Statist. Math.*, **48**, 157-168.

Ferguson, T.S. (1967): On characterizing distributions by properties of order statistics. *Sankhyā, Ser. A*, **29**, 265-278.

Franco, M. and Ruiz, J.M. (1995): On characterization of continuous distributions with adjacent order statistics. *Statistics*, **26**, 375-385.

Franco, M. and Ruiz, J.M. (1997): On characterizations of distributions by expected values of order statistics and record values with gap. *Metrika*, **45**, 107-119.

Franco, M. and Ruiz, J.M. (1999): Characterization based on conditional expectations of adjacent order statistics: a unified approach. *Proc. Amer. Math. Soc.*, **127, no.3**, 861-874

Galambos, J. (1978, 1987): The asymptotic theory of Extreme Order Statistics. *Wiley, New York* **(1st ed.)**. *Kreiger, FL* **(2nd ed.)**

Galton, F. (1902): The most suitable proportion between the values of first and second prized. *Biometrika*, **1**, 385-390.

Godwin, H.J. (1949): Some low moments of order statistics. *Ann. Math. Statist.*, **20**, 279-285.

Govindrajulu, Z. (1963): On moments of order statistics and quasi-ranges from normal population. *Ann. Math. Statist.*, **34**, 633-651.

Govindrajulu, Z. (1975): Characterizations of the exponential distribution using lower moments of order statistics. *In Statistical Distributions in Scientific Work*, **Vol.3**, 117-129 (Editor: G.P. Patil; S. Kotz and J.K. Ord).

Gumbel, E.J. (1958): Statistics of Extremes. *Columbia University Press, New York.*

Hamdan, M.A. (1972): On characterization by conditional expectation. *Technometrics*, **14**, 497-499.

Harter, H.L. (1978-1992): The Chronological Annotated Bibliography of Order Statistics. **Vol. 1-8**, *American Sciences Press, Columbus, Ohio.*

Huang, J.S. (1989): Moments problem of order statistics: A review. *Inter. Statist. Rev.*, **57**, 59-66.

Hwang, J.S. and Lin, G.D. (1984): Characterizations of distributions by linear combinations of moments of order statistics. *Bull. Inst. Math., Academia Sincia*, 12, 179-202.

Jones, H.L. (1948): Exact lower moments of order statistics in small samples from a normal distribution. *Ann. Math. Statist.*, **19**, 270-273.

Joshi, P.C. (1971): Recurrence relations for the mixed moments of order statistics. *Ann. Math. Statist.*, **42**, 1096-1098.

Joshi, P.C. and Balakrishnan, N. (1982): Recurrence relations and identities for the product moments of order statistics. *Sankhyā, Ser. B*, **44**, 39-49.

Kamps, U. (1991): A general recurrence relations for moments of order statistics in a class of probability distributions and characterizations. *Metrika*, **38**, 215-225.

Khan, A.H.; Yaqub, M. and Parvez, S. (1983a): Recurrence relations between moments of order statistics. *Naval Res. Logist. Quart.*, **30**, 419-441. Corrections **32** (1985), 693.

Khan, A.H., Parvez, S. and Yaqub, M. (1983b): Recurrence relations between product moments of order statistics. *J. Statist. Plan. Infer.*, **8**, 175-183.

Khan, A.H. and Khan, I.A. (1986): Characterizations of the Pareto and the power function distributions. *J. Statist. Res.*, **20**, 71-79.

Khan, A.H. and Ali, M.M. (1987): Characterization of probability distribution through higher order gap. *Commun. Statist.-Theory Meth.*, **16**, 1281-1287.

Khan, A.H. and Beg, M.I. (1987): Characterization of the Weibull distribution by conditional variance. *Sankhyā, Ser. A.*, **49**, 268-271.

Khan, A.H. and Khan, I.A. (1987): Moments of order statistics from Burr distribution and its characterization. *Metron*, **45**, 21-29.

Khan, A.H. and Abu-Salih, M.S. (1988): Characterization of Weibull and inverse Weibull distributions through conditional moments. *J. Internat. Opt. Sc.*, **9**, 355-362.

Khan, A.H.; Khan, R.U. and Samad, A. (1988): Moments of order statistics from a logistic distribution and a characterization theorem. *Estadistica*, **40**, 27-35.

Khan, A.H. and Abu-Salih, M.S. (1989): Characterization of probability distributions by conditional expectation of order statistics. *Metron*, **47**, 171-181.

Khan, A.H. (1991): A note on relation between binomial and negative binomial sums. *Aligarh Journal of Statistics*, 91-92.

Khan, A.H. and Abouammoh, A.M. (2000): Characterizations of distributions by conditional expectation of order statistics. *J. Appl. Statist. Sc.*, **9**, 159-167.

Khan, A.H. and Athar, H. (2000a): On characterization of distributions by conditioning on a pair of order statistics. *Submitted for publication.*

Khan, A.H. and Athar, H. (2000b): Characterization of distributions through order statistics. *Submitted for publication.*

Khan, A.H. and Athar, H. (2000c): Characterization of probability distributions through conditional expectation of function of two order statistics. *Submitted for publication.*

Khan, A.H. and Athar, H. (2000d): On characterization of probability distributions through conditional expectation of order statistics. *Submitted for publication.*

Kotlarski, I. I. (1972): On a characterization of some probability distributions by conditional expectations. *Sankhyā*, *Ser. A*, **34**, 461-466.

Lehman, E.L. (1959): Testing Statistical Hypothesis. *John Wiley and Sons, New York.*

Lin, G.D. (1988): Characterizations of diostributions via relationships between two moments of order statistics. *J. Statist. Plan. Infer.*, **19**, 73-80.

Lin, G.D. (1989): The product moments of order statistics with applications to characterizations of distributions. *J. Statist. Plan. Infer.*, **21**, 395-406.

Lloyd, E. H. (1952): Least square estimation of location and scale parameters using order statistics. *Biometrika*, **39**, 88-95.

Ouyang, L.Y. (1983): On characterizations of distributions by conditional expectations. *Tamkang J. Management Sci.*, **4**, 13-22.

Ouyang, L.Y. (1995): Characterization through the conditional distribution of a function of one order statistic relative to an adjacent one. *Sankhyā, Ser. A*, **57**, 500-503.

Pakes, A.G.; Fakhry, M.E.; Mahmoud, M.R. and Ahmad, A.A. (1996): Characterizations by regression of order statistics. *J. Appl. Statist. Sc.*, **3**, 11-23.

Pearson, K. (1902): Note on Francis Galton's difference problem. *Biometrika*, **1**, 390-399.

Pearson, K. (1934, 1955): Tables of Incomplete Beta Function. *Cambridge University Press, England.*

Sarhan, A.E. and Greenberg, B.G. (Eds.) (1962a): Contributions to Order Statistics. *Wiley, New York.*

Shanbhag, D.N. (1970): The characterizations of exponential and geometric distributions. *J. Amer. Statist. Assoc.*, **65**, 1256-1259.

Sillitto, G.P. (1951): Interrelations between certain linear systematic statistics of samples from any continuous population. *Biometrika*, **38**, 377-382.

Sillitto, G.P. (1964): Some relations between expectations of order statistics in samples of different sizes, *Biometrika*, **51**, 259-262.

Talwalker, S. (1977): A note on characterization by the conditional expectation. *Metrika*, **24**, 129-136.

Wu, J.W. and Ouyang, L.Y. (1996): On characterizing distributions by conditional expectations of the function of order statistics. *Metrika*, **43**, 135-147.

Johnson, N.L. and Kotz, S. (1970): Distribution in Statistics: Continuous Univariate Distributions, Vol. I & II, John Wiley & Sons, New York.